A Lexicographic Algorithm Solving a Problem of a Multiobjective Flow in a Network*

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1. Introduction

Let for a given oriented connected network \( G(N, U) \) with a set of nodes \( N=\{1, \ldots, n\} \), \( l=s \) and \( n=t \), and a set of arcs \( A=\{(i, j): i, j \in N\} \), \( x_{ij} \) denotes the flow along the arc \((i, j)\) and \( x \) the flow vector. The problem for a multiproduct flow (MF) in \( G \) could be formulated as follows:

\[
\text{MF: } \min f_1(X) = \sum_{(i, j) \in U} a_{ij} x_{ij},
\]

\[
\ldots
\]

\[
\text{min } f_k(X) = \sum_{(i, j) \in U} a_{ij} x_{ij}
\]

under the constraints

\[
\sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} v, & i = s, \\ 0, & i \neq s, t, \\ -v, & i = t \end{cases}
\]

(1)

\[
0 \leq x_{ij} \leq c_{ij}, \quad (i, j) \in U
\]

(2)

where \( c_{ij} \) is the upper limit of the flow along the arc \((i, j)\).

The constraints (1) are called constraints for flow conservation, and constraints (2) - capacity constraints. The problem is a partial case of the multicriteria linear problem.

It is obvious that the optimal solution of the problem, which is multicriteria, cannot be found by the algorithms designed to find a minimal flow in the single criterion case. These algorithms are efficient in a computing aspect mainly for large dimension problems. This efficiency designates the necessity for neglecting the general methods of multicriteria linear problems solution and the search for specialized algorithms using the flow structure of the constraints matrix of the problem being set.

The problem of such algorithms design is not at all easy, since the solution of

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the multicriteria problems is connected with the addition of supplementary constraints
to the constraints of the problem considered, which destroys the special structure of
the constraints matrix.

Some algorithms, adaptations of simplex-methods have been proposed, that use
the presence of flow constraints in the constraints matrix \([1, 2]\). A specialized
multicriteria simplex-algorithm has been proposed in \([3]\) for generation of all non-
dominated solutions of a MF. A methodology is developed in \([4]\) for the generation of
a set of non-dominated solutions of the bicriteria flow problem. The properties of the
non-dominated solutions of a MF have been investigated in \([5]\).

An approach solving a MF is recommended in the present paper, characterized
by apriori criteria ranking in conformance with the decision maker’s (DM’s) prefer-
ences. In this case a sequence of single criterion flow problems is solved, the network
structure being altered in each one in such a way, that the value of the criterion with
higher priority is not deteriorated. An analogue of the out-of kilter method is repre-
sented in \([6]\), satisfying the conditions for flow conservation, while those for capacity
duality may be broken. The structure of the network for a subsequent single-
criterion problem is defined in \([7]\) depending on the dual conditions being satisfied.

2. A lexicographic algorithm

When defining the MF problem, the value of the variable \(v\) should be specified – the
flow size. Three cases are possible: a maximal flow is searched for; a flow with a
defined size is searched for; an optimal size flow is seeked for. In the last case an arc
\((t, s)\) is added to the network arcs set and the condition that the balance of the flow
should be 0 at nodes \(s\) and \(t\) also, is implied.

It is assumed that the DM has ranked the objective functions according to their
absolute value (priority). In general the algorithm consists of the following: a single-
criterion optimization problem for a minimal flow in a network is solved with the
highest priority criterion, and a network – the network being set. Let the flow – a
solution of this problem, be denoted by \(x\). A constraint is set that this criterion re-
mains equal to its minimum and under this condition the second in significance crite-
ron is minimized, after that – the third, under the condition that the first two will
preserve their values and so on, until all the criteria are minimized.

The direct addition of these constraints would deconstruct the flow structure of
the problem constraints. Hence this property of the optimal flow is used that in an
appropriately built network \(G(x)\), the structure of which depends on the values of the
flow \(x\) along the original network, there do not exist cycles of a negative value. The
omitting of an additional flow of cycles with a positive value would worsen the flow
value, that is why all the arcs forming cycles with a positive value are extracted from
the given network \(G\), i.e. the network \(G\) is reduced to the network \(G_{i}\) so that in the
network \(G_{i}(x)\) there are cycles with a zero value only. On the network thus obtained
the problem for a minimal flow is solved with a criterion – the second preferred func-
tion, a new network is constructed, depending on its optimal solution, the arcs forming
positive cycles are isolated and so on, till all the criteria are minimized or a graph not
connected with respect to the node \(s\) and \(t\) is constructed.

The general description of the algorithm is the following:

\textbf{Iteration 0}
\[ p=1 \text{ and } U=U \text{ is set.} \]
Iteration $p$

the problem for a minimal flow

$$\text{Min}\ F = \min_{(i, j) \in U} \sum_{i, j} a_{ij} x_{ij}$$

is solved, where $a_{ij} = a_{ij}^p$

subject to constraints

$$\sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} v_i & i = s, \\ 0 & i \neq s, t, \\ -v_j & i = t, \end{cases}$$

$$0 \leq x_{ij} \leq c_{ij}, \quad (i, j) \in U.$$
The criteria satisfy the following:

\[ f_1(X) = x_{13} + 3x_{35} + 3x_{45} + 2x_{56} + 2x_{68}, \]
\[ f_2(X) = 4x_{13} + 4x_{25} + 2x_{56} + 7x_{68} + 3x_{16}, \]
\[ f_3(X) = 2x_{23} + 4x_{25} + 7x_{37} + 4x_{45} + 8x_{46} + 6x_{57} + 6x_{16}, \]
\[ f_4(X) = 2x_{12} + 4x_{13} + 7x_{25} + 2x_{35} + 2x_{37} + 4x_{45} + 10x_{56} + 2x_{57} + 9x_{16} + 2. \]

The optimal solution of the problem with respect to the criterion \( f_1(X) \) is found at the first iteration. The values of the flow are given below, \( s=1 \) and \( t=8 \):

- \( x_{12} = 9, x_{13} = 10, x_{25} = 0, x_{24} = 3, x_{36} = 0, x_{35} = 6, x_{37} = 7, x_{45} = 3, x_{46} = 1, x_{47} = 2, \)
- \( x_{56} = 0, x_{57} = 8, x_{59} = 15, x_{68} = 7, x_{84} = 4, x_{85} = 19. \)

The arcs \((1, 3), (2, 3), (3, 7), (4, 5), (5, 6), (6, 8)\) are isolated and on the network obtained the second criterion is optimized. The values of the flow are as follows:

- \( x_{12} = 11, x_{13} = 10, x_{25} = 0, x_{24} = 5, x_{36} = 0, x_{35} = 6, x_{37} = 7, x_{45} = 3, x_{46} = 4, x_{47} = 2, \)
- \( x_{56} = 0, x_{57} = 8, x_{59} = 15, x_{68} = 5, x_{84} = 6, x_{85} = 21. \)

The arcs \((2, 6), (2, 4)\) are isolated and the third criterion is optimized on the newly constructed network. The flow values are:

- \( x_{12} = 11, x_{13} = 10, x_{25} = 0, x_{24} = 5, x_{36} = 0, x_{35} = 6, x_{37} = 7, x_{45} = 3, x_{46} = 4, x_{47} = 2, \)
- \( x_{56} = 0, x_{57} = 8, x_{59} = 15, x_{68} = 5, x_{84} = 6, x_{85} = 21. \)

The arc \((2, 5)\) is isolated. A network not connected to the source and the sink is obtained. The solution searched for is attained.

3. Conclusion

An algorithm is suggested that implements a lexicographic approach in the solution of the problem for a multiobjective flow in a network, when the criteria are ranked by priority. At each subsequent iteration the next criterion with a lower priority in the ranking order is minimized on a network obtained from the initial one after isolation of the arcs defined at the previous iteration. In this way the possibility to apply efficient flow algorithms is conserved.

The algorithm correctness is proved and it is applied on a test example.
References


Лексикографический алгоритм решения задачи многокритериального потока в сети

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Исследована задача многокритериального потока в сети в случае, когда критерии даны в приоритете ряду. Для решения задачи предлагается алгоритм, реализующий лексикографический подход, при чем на каждой итерации решается однокритериальная задача минимального потока в сети.