Generalizations of Some Criteria for P-Matrices and M-Matrices

Vladimir V. Monov

Institute of Information Technologies, 1113 Sofia

1. Introduction

It is well known that the set of P-matrices includes several important classes of matrices such as M-matrices and positive definite matrices. P-matrices arise in various theoretical and applied fields, for example in linear complementarity theory [9], in the analysis of the solution set of systems of linear interval equations [6], in the study of convex sets of matrices [5]. The class of M-matrices is characterized by a special sign pattern of matrix elements which suggests relations with the theory of nonnegative matrices. From an application point of view, M-matrices play an important role in certain economic models [8] and provide a tool for stability analysis of composite dynamical systems [1].

There is a large number of different in form but essentially equivalent conditions that are necessary and sufficient for a given matrix to be a P-matrix or an M-matrix. Selected lists of such conditions together with the relevant theory are given in [4]. Some generalizations of the P-matrix concept related mainly to the linear complementarity problems can be found in [3], [9] and their references. The results in [7] present criteria for the P-property of all matrices belonging to an interval matrix set and introduce the notion of interval P-matrices.

In this paper, we study the P-property and M-property of matrices which are elements of compact and convex matrix sets. Our aim is to establish criteria characterizing these properties with respect to all elements of the matrix set. In Theorems 2.1 and 3.1, we have obtained general criteria which are valid for any compact convex set of matrices. Several special cases of compact convex sets found in the literature are also considered. In each case, in addition to the general criterion, we have derived equivalent necessary and sufficient conditions which provide a finite test for the P-property and M-property of all matrices belonging to the matrix set. The obtained results are based on well known criteria for a single P-matrix and M-matrix and generalize these criteria to the case of compact and convex matrix sets.
Let $\mathbb{R}^n$ and $\mathbb{M}(\mathbb{R})$ be the normed vector spaces of $n$-dimensional vectors and $n \times n$ matrices with real elements, respectively. The usual inner product in $\mathbb{R}^n$ will be denoted by $(.,.)$, i.e. $(x, y) = x^T y$, $x, y \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$ ($A \in \mathbb{M}(\mathbb{R})$) we shall write $x \geq 0$ ($A \succeq 0$) if all elements of $x(A)$ are nonnegative. In this case, $x(A)$ is said to be nonnegative. The positivity of a vector and matrix is defined in a similar way.

A matrix $A \in \mathbb{M}(\mathbb{R})$ is called a P-matrix if all $k \times k$ principal minors of $A$ are positive for $k=1, \ldots, n$. The set of all P-matrices will be denoted by $P_n(\mathbb{R})$. It is obvious that $P_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$, where $GL_n(\mathbb{R}) \subseteq \mathbb{M}(\mathbb{R})$, is the set of nonsingular matrices. The following well known characterizations of a P-matrix [4] will be used in our results.

Given $A \in P_n(\mathbb{R})$, each of the next conditions is necessary and sufficient for $A \in P_n(\mathbb{R})$:

1. For every nonzero $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ there is an index $i \in \{1, \ldots, n\}$ such that $x_i(Ax_i) > 0$. 
2. For every nonzero $x \in \mathbb{R}^n$ there is some nonnegative diagonal matrix $D = D(x) \in \mathbb{M}_n(\mathbb{R})$ such that $x^T (D(x)A)x > 0$.
3. For every nonzero $x \in \mathbb{R}^n$, there is some positive diagonal matrix $D = D(x) \in \mathbb{M}_n(\mathbb{R})$ such that $x^T (D(x)A)x > 0$.

If $A \in P_n(\mathbb{R})$ then it is also well known that $A^T \in P_n(\mathbb{R})$ and $A+xD, DA, AD \in P_n(\mathbb{R})$ for every diagonal matrix $D$ with positive diagonal elements.

The next theorem characterizes the P-property of matrices belonging to a compact convex set of matrices.

**Theorem 2.1.** Let $K \subseteq \mathbb{M}(\mathbb{R})$ be a compact convex set and let $\varepsilon$ be the set of its extreme points. The following conditions are equivalent:

1. $K \subseteq P_n(\mathbb{R})$
2. for every nonzero $x \in \mathbb{R}^n$, there is a nonnegative diagonal matrix $D = D(x) \in \mathbb{M}_n(\mathbb{R})$ such that $x^T (D(x)A)x > 0$ for all $A \in K$.
3. for every nonzero $x \in \mathbb{R}^n$, there is some nonnegative diagonal matrix $D = D(x) \in \mathbb{M}_n(\mathbb{R})$ such that $x^T (D(x)A)x > 0$ for all $A \in \varepsilon$.

**Proof.** Since $K$ is compact and convex, it is obvious that (ii) and (iii) are equivalent. Also, (ii) implies (i) by criterion (P2). Hence it remains to show that (i) implies (ii). First, we shall prove this for a polytope of matrices, i.e. $K = \text{convex hull} \{A_1, \ldots, A_m\}$ where $A_i \in M_n(\mathbb{R})$, $i = 1, \ldots, m$ are given matrices. Let $D \subseteq \mathbb{M}_n(\mathbb{R})$ be defined as $D = \{D \in \text{convex hull} \{d_1, \ldots, d_n\} : d \geq 0, d \in \{1, \ldots, n\}\}$. Proceeding by contradiction, assume that (i) holds and suppose that for some $x \in \mathbb{R}^n$, $x \neq 0$, there is no matrix $D \in \varepsilon$ satisfying (ii). Let $S_2 \subseteq \mathbb{R}^n$ be the set defined by:

3. $S_2 = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : z_i = x^T A_i x_i, i = 1, \ldots, m, D \in \Delta\}$,

and $S_2$ be the positive orthant in $\mathbb{R}^n$, i.e:

4. $S_2 = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n : z_i > 0, i = 1, \ldots, m\}$.

It is easily seen that $S_2$ is a closed convex cone in $\mathbb{R}^n$. By our assumption there is no $D \in \Delta$ satisfying (ii) and therefore $S_1 \cap S_2 = \emptyset$. Since both $S_1$ and $S_2$ are nonempty convex
sets, this implies that there exists a hyperplane in $\mathbb{R}^n$ separating $S$ and $S'$, i.e. there exists a nonzero vector $a = (a_1, \ldots, a_n)^t \in \mathbb{R}^n$ and real number $\alpha$ such that

$$a \cdot z = a_1 z_1 + \ldots + a_n z_n \leq \alpha \text{ for all } z \in S,$$

$$a \cdot z = a_1 z_1 + \ldots + a_n z_n \geq \alpha \text{ for all } z \in S'.$$

Inequality (6) implies that $\alpha_i \geq 0$, $i=1, \ldots, n$. Indeed, if some $\alpha_i < 0$, then there always exist $z \in S'$ for which (6) is violated. Similarly, if $\alpha > 0$, then one can find $z \in S$ such that (6) is not satisfied. In view of (3), inequality (5) then becomes

$$x^T \beta x \leq 0 \text{ for all } \beta \in \mathcal{K},$$

where $\beta = (\beta_1, \ldots, \beta_m)^t$, $i=1, \ldots, m$. Hence $(\beta_1 A_1 + \ldots + \beta_m A_m) \in \mathcal{K}$ by criterion (P3) instead of (P2). This contradicts (i).

Now, let $\mathcal{K}$ be any compact convex set of matrices for which condition (i) holds. Then one can find a closed convex polytope $K'$ such that $K \subseteq K' \subseteq \mathcal{P}$. The proof for polytopes shows that for every $x \in \mathbb{R}^n$, $x \neq 0$, there is a nonnegative diagonal matrix $D(x)$ satisfying $x^T D(x) A x > 0$ for all $A \in K'$. The inclusion $K \subseteq K'$ then implies that $x^T D(x) A x > 0$ for all $A \in K$, which completes the proof.

Theorem 2.1 is a generalization of criterion (P2) where condition (iii) of this theorem gives an extreme point characterization for the $\mathcal{P}$-property of every $A \in K$. It is easily seen that both conditions (ii) and (iii) can be analogously formulated using criterion (P3) instead of (P2). In this case, we obtain that $K \subseteq \mathcal{P}$ if and only if for every nonzero $x \in \mathbb{R}^n$ there is some positive diagonal matrix $D = D(x) \in M_n(\mathbb{R})$, satisfying (1) or (2). Although (P1), (P2) and (P3) are equivalent for a single matrix $A \in \mathcal{M}_n(\mathbb{R})$, the next simple example shows that criterion (P1) cannot be generalized in a similar way.

**Example 2.2.** Let $K = \text{convex hull} \{A, B\}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

It can be easily shown that condition (iii) of Theorem 2.1 is satisfied for every $x = (x_1, x_2)^t \in \mathbb{R}^2$, $x \neq 0$. Indeed, if $x_1 = 0$, we take $D(x) = \text{diag}(0, 1)$, so that $x^T D(x) A x = x_2^2 > 0$. If $x_1 \neq 0$, then without loss of generality we can consider real vectors $x \in \mathbb{R}^2$ in the form $x = (1, x_2)^t$. In this case, if $x_2 > 1$ again $D(x) = \text{diag}(1, 1)$ is an appropriate matrix satisfying $x^T D(x) A x = x_2^2 > 0$ and $x^T D(x) B x = x_2^2 (x_2 - 1) > 0$. Finally, if $x_2 \leq 1$, we can choose $D(x) = \text{diag}(1, x_2)$ so that $x^T D(x) A x = x_2^2 (x_2 - 1) > 0$. Hence, by Theorem 2.1, every matrix in $K$ is a $\mathcal{P}$-matrix. However, the inclusion $K \subseteq \mathcal{A}$ does not imply that for every nonzero $x = (x_1, x_2)^t \in \mathbb{R}^2$, there is some index $i \in \{1, 2\}$ such that $x_A(x_i) > 0$ and $x_B(x_i) = 0$. In fact, for $x = (1, 1)^t$ we have $x_A(1) = x_B(1) = 0$, and $x_A(1) = x_B(1) = 0$.

In the rest of this section, we study the $\mathcal{P}$-property of several special cases of compact convex sets of matrices. Let $\mathcal{O}$ be the matrix whose elements are all equal to 1 and let $\circ$ denotes the Hadamard (elementwise) product of matrices. For any $A, B \in \mathcal{M}_n(\mathbb{R})$ the following sets are defined in [5]:

$$K_1 = \{C(t) : C(t) = tA + \mathcal{O}, \mathcal{O} \in [0, 1]\},$$

$$K_2 = \{C(t) : C(t) = tA + (1 - t) B, \mathcal{O} \in \text{diag} (t_1, \ldots, t_n), t_i \in [0, 1]\},$$

$$K_3 = \{C(t) : C(t) = tA + B (1 - t), \mathcal{O} \in \text{diag} (t_1, \ldots, t_n), t_i \in [0, 1]\},$$

$$K_4 = \{C(t) : C(t) = tA + B (1 - t), \mathcal{O} \in \text{diag} (t_1, \ldots, t_n), t_i \in [0, 1]\}.$$
nonsingularity criterion \((12)\). Conversely, if 
\[
\det(a) \neq 0 
\]
where \(\sigma\) denotes the set of eigenvalues (spectrum) of \((\cdot)\). Criteria for nonsingularity of matrices in... and... are obtained in [5] as follows:

(12) \(K_n \subset \text{GL}_n(\mathbb{R}),\) if and only if \(\sigma(B, \lambda^{-1}) \cap (-\infty, 0) = \emptyset,\)

where \(\sigma(\cdot)\) denotes the set of eigenvalues (spectrum) of \((\cdot)\). Criteria for nonsingularity of matrices in... and... are obtained in [5] as follows:

(13) \(K_i \subset \text{GL}_i(\mathbb{R}),\) if and only if \(BA^{-1} \in P_n(\mathbb{R}),\)

(14) \(K_c \subset \text{GL}_c(\mathbb{R}),\) if and only if \(BA^{-1} \in P_n(\mathbb{R}).\)

It can be easily seen that \((11)\) is an interval matrix set and in this case, various criteria for \(K \subset \text{GL}_n(\mathbb{R}),\) are given in [6]. Furthermore, necessary and sufficient conditions for \(K \in P_n(\mathbb{R})\) are obtained in [7]. In what follows, we establish criteria for \(K_n \in P_n(\mathbb{R}),\)

\textbf{Theorem 2.3.} Let \(AB \in M_n(\mathbb{R}).\) The following conditions are equivalent:

(a) \(K_n \in P_n(\mathbb{R});\)

(b) for every nonzero \(x \in \mathbb{R}^n\) there is some nonnegative diagonal matrix \(D = D(x) \in P_n(\mathbb{R})\) such that \(x^t(D(x)A)x > 0\) and \(x^t(D(x)B)x > 0;\)

(c) \(AB \in P_n(\mathbb{R})\) and

\[
\sigma(B_{\alpha\alpha}(A_{\alpha\alpha}^{-1})) \cap (-\infty, 0) = \emptyset \text{ for every } \alpha \subseteq \{1, \ldots, n\},
\]

where \(A_{\alpha\alpha}\) and \(B_{\alpha\alpha}\) are principal submatrices of \(A\) and \(B\) corresponding to the pair of indices \(\alpha(\cdot)\).

\textit{Proof.} The equivalence of (a) and (b) follows from Theorem 2.1. It will be shown that (a) \(\leftrightarrow\) (c). Let \(\alpha \subseteq \{1, \ldots, n\}.\) If \(K \in P_n(\mathbb{R}),\) then obviously \(A, B \in P_n(\mathbb{R})\) and

\[
\det C = \det (tA_{\alpha\alpha} + (1-t)B_{\alpha\alpha}) > 0 \text{ for every } t \in [0, 1],
\]

which implies (15) by the nonsingularity criterion (12). Conversely, if \(A \in P_n(\mathbb{R})\) and for \(t = 1\) we have

(16) \(\det C_{\alpha\alpha}(t) = \det A_{\alpha\alpha} > 0,\)

and from (15), it follows that

(17) \(\det C_{\alpha\alpha}(t) = \det (tA_{\alpha\alpha} + (1-t)B_{\alpha\alpha}) \neq 0 \text{ for every } t \in [0, 1].\)

Since \(\det C_{\alpha\alpha}(t)\) depends continuously on \(t,\) conditions (16) and (17) imply that \(\det C_{\alpha\alpha}(t) > 0\) for every \(t \in [0, 1],\) i.e. \(C(\cdot)\) is a \(P\)-matrix for all values of...

\textbf{Theorem 2.4.} Let \(A, B \in M_n(\mathbb{R})\) and let \(\varepsilon \subseteq K_n\) be the set defined as

\[
\varepsilon = \{(C(t) : C(t) = TA + (I-T)B, T = \text{diag}(t_1, \ldots, t_n), t_i \in [0, 1])\}
\]

Then the following conditions are equivalent:

(a) \(K_n \subset P_n(\mathbb{R});\)

(b) for every nonzero \(x \in \mathbb{R}^n\) there is some nonnegative diagonal matrix
\( D = D(x) \in M_n(\mathbb{R}) \) such that \( x' (D(x) C(t)) x > 0 \) for all \( C(t) \in \varepsilon_i; \)

(c) for every nonzero \( x \in \mathbb{R}^n \) there is an index \( i \in [1, \ldots, n] \) such that \( x_i (C(t)x) > 0 \) for all \( C(t) \in \varepsilon_i; \)

(d) \( AB \in P_n(\mathbb{R}) \) and

\[
B = \{ A_i \}^{-1} \in P_n(\mathbb{R}) \text{ for every } \alpha \subseteq \{1, \ldots, n\},
\]

where \( |\alpha| \) denotes cardinality of \( \alpha; \)

(e) \( \varepsilon_i \subseteq P_n(\mathbb{R}) \).

**Proof.** (a)\( \Leftrightarrow \) (b) Follows from Theorem 2.1 by noting that \( K = \text{convex hull } (\varepsilon_i). \)

(a)\( \Leftrightarrow \) (d). As in Theorem 2.3, the proof is based on the fact that (18) is equivalent to

\[
\det C_{\alpha}(t) = \det (T_{\alpha} A_{\alpha} + (I - T_{\alpha}) B_{\alpha}) \neq 0 \text{ for every } T = \text{diag } (t_i, \ldots, t_n), \ t_i \in [0, 1] \text{ and that } \det C_{\alpha}(t) \text{ depends continuously on } t_i, i = 1, \ldots, n.
\]

(a)\( \Leftrightarrow \) (e). Since \( \varepsilon_i \subseteq K_{\alpha} \), it is obvious that (a)\( \Leftrightarrow \) (e). To prove the converse implication, for any \( x \in \mathbb{R}^n \) we define \( C_{\alpha} \in \varepsilon_i \) as

\[
C_{\alpha} = TA + (I - T) B \text{, } T = \text{diag } (t_i, \ldots, t_n)
\]

where

\[
t_i = \begin{cases} 1, & \text{if } x_i (Ax) \leq x_i (Bx) \\ 0, & \text{if } x_i (Ax) > x_i (Bx) \end{cases}, \quad i = 1, \ldots, n.
\]

Then for every \( C(t) \in K_{\alpha} \) and every \( i \in \{1, \ldots, n\} \), we have

\[
x_i (C(t)x) = t_i x_i (Ax) + (1 - t_i) x_i (Bx) \geq \min \{x_i (Ax), x_i (Bx)\} = x_i (C_{\alpha}x)
\]

If \( \varepsilon_i \subseteq P_n(\mathbb{R}) \) and \( x \neq 0 \), then \( C_{\alpha} \in \varepsilon_i \) is a \( P \)-matrix and by criterion (P1) there exists some \( i \in \{1, \ldots, n\} \), such that \( x_i (C_{\alpha}x) > 0 \). Inequality (21) now implies

\[
x_i (C(t)x) \geq x_i (C_{\alpha}x) > 0 \text{ for all } C(t) \in K_{\alpha},
\]

and hence, we obtain that \( K_{\alpha} \subseteq P_n(\mathbb{R}) \).

(c)\( \Leftrightarrow \) (e). If condition (c) holds, then \( \varepsilon_i \subseteq P_n(\mathbb{R}) \) by criterion (P1). The converse implication follows from inequality (22) where \( C_{\alpha} \in \varepsilon_i \) is given by (19) and (20). This completes the proof.

In a similar way, we obtain the following necessary and sufficient conditions for \( K_{\alpha} \subseteq P_n(\mathbb{R}) \).

**Theorem 2.5.** Let \( AB \in M_n(\mathbb{R}) \) and let \( \varepsilon_i \subseteq K_{\alpha} \) be the set defined as

\[
\varepsilon_i = \{ C(t) : C(t) = AT + B(I - T), \ T = \text{diag } (t_i, \ldots, t_n), \ t_i \in (0, 1) \}.
\]

Then the following conditions are equivalent:

(a) \( K_{\alpha} \subseteq P_n(\mathbb{R}) \);

(b) for every nonzero \( x \in \mathbb{R}^n \) there is some nonnegative diagonal matrix \( D = D(x) \in M_n(\mathbb{R}) \) such that \( x' (D(x) C(t)) x > 0 \) for all \( C(t) \in \varepsilon_i; \)

(c) for every nonzero \( x \in \mathbb{R}^n \) there is an index \( i \in \{1, \ldots, n\} \) such that \( x_i (C(t)x) > 0 \) for all \( C(t) \in \varepsilon_i; \)

(d) \( AB \in P_n(\mathbb{R}) \) and
\[(B_{\infty})^{-1} A_{\infty} \in P_{\infty} (\mathbb{R}) \] for every \( \alpha \subseteq \{1, \ldots, n\}, \)

where \(|\alpha|\) denotes cardinality of \(\alpha\);

(e) \( e_c \subset P_n (\mathbb{R}) \).

Note that \(e_c\) and \(e_r\) in Theorems 2.4 and 2.5 are the sets of extreme points of \(K_c\) and \(K_r\) respectively. Conditions (e) in both theorems show that \(K_c \subset P_n (\mathbb{R})\) (respectively, \(K_r \subset P_n (\mathbb{R})\)) if and only if the extremematrixes of \(K_c\) (respectively, \(K_r\)) are P-matrices. Since the cardinality of \(e_r\) is \(2^n\), these conditions provide a finite test for the P-property of every matrix in \(K_c\) or \(K_r\). Condition (c) in Theorem 2.3 and conditions (d) in Theorems 2.4 and 2.5 are essentially based on the nonsingularity criteria (12), (13) and (14), respectively. These conditions also lead to a finite characterization of the P-property of matrices in \(K_c\) and \(K_r\). Finally, conditions (c) in Theorems 2.4 and 2.5 show that, for a certain type of convex matrix sets, criterion (P1) still can be generalized analogously as (P2) and (P3). A similar conclusion follows from [7] where the P-property of interval matrices is characterized in terms of condition (P1).

Example 2.6. Let \(K_c, K_r\) and \(K_{\alpha}\) be given by (8), (9) and (10), respectively, where the values of \(A\) and \(B\) are taken from [5] as follows

\[
A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.
\]

For every \(\alpha \subseteq \{1, 2, 3\}\), it can be easily checked that \(\sigma((B_{\infty})^{-1} A_{\infty}) \cap (-\infty, 0] = \emptyset\), and hence by condition (c) of Theorem 2.3 \(K_c \subset P_n (\mathbb{R})\). Also, every extremematrix of \(K_{\alpha}\) is a P-matrix and by Theorem 2.4 \(K_c \subset P_n (\mathbb{R})\). However, for \(T = \text{diag}(1, 0, 0)\), we have

\[
C = AT + B(I-T) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix},
\]

which is an extreme point of \(K_c\) and obviously \(\det C = 0\). Thus, \(K_c \subset P_n (\mathbb{R})\) and \(K_{\alpha} \subset GL_n (\mathbb{R})\).

3. Sets of M-matrices

Let \(Z_n (\mathbb{R}) \subset P_n (\mathbb{R})\) be the set defined by

\[Z_n (\mathbb{R}) = \{A = (a_{ij}) \in M_n (\mathbb{R}) : a_{ij} \leq 0 \text{ if } i \neq j, i, j = 1, \ldots, n\} .\]

The elements of \(Z_n (\mathbb{R})\) are related with the nonnegative matrices by the following representation: \(A \in Z_n (\mathbb{R})\) if and only if \(A = a \alpha I - P\) for some \(a \in \mathbb{R}\) and some \(P \in M_n (\mathbb{R})\) with \(P \geq 0\).

A matrix \(A \in M_n (\mathbb{R})\) is called an M-matrix if \(A \in Z_n (\mathbb{R})\) and the eigenvalues of \(A\) all have positive real parts. The set of M-M-matrices will be denoted by \(M_n (\mathbb{R})\). A well known criterion [4] states that \(A \in M_n (\mathbb{R})\) if and only if \(A \in Z_n (\mathbb{R})\) and all \(k \times k\) principal minors of \(A\) are positive for \(k = 1, \ldots, n\). Thus, we have

\[
M_n (\mathbb{R}) = Z_n (\mathbb{R}) \cap P_n (\mathbb{R}) .
\]

In view of the special structure of M-matrices (\(M_n (\mathbb{R}) \subset Z_n (\mathbb{R})\)), criterion (P2) from the previous section can be specialized to the following necessary and sufficient condition.

\(A \in Z_n (\mathbb{R})\) is an M-matrix if and only if for every nonzero \(x \in \mathbb{R}^n\) with \(x \geq 0\) there is
some nonnegative diagonal matrix $D = D(x) \in M_\mathbb{R}$ such that

$$(24) \quad x^T (D(x) A) x > 0.$$  

If $A \in M_\mathbb{R}$ then the inclusion $M_\mathbb{R} \subseteq P_\mathbb{R}$ implies (24) by (P2). To show that (24) is also sufficient, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $|x|$ denote $|x| = (|x_1|, \ldots, |x_n|)^T$. Then for every $A \in Z_\mathbb{R}$ and every nonnegative diagonal matrix $D$, we have

$$(25) \quad x^T (D(x) A) x \geq |x|^T (\alpha D - DP) |x| = |x|^T (DA) |x|.$$  

In (25), $A$ is represented as $A = \alpha I - P$ where $\alpha \in \mathbb{R}$ and $P \geq 0$ and the inequality follows from the fact that $DP \geq 0$ which implies $x^T (DP) x \leq |x|^T (DP) |x|$. If condition (24) holds, then for every nonzero $x$ there is a nonnegative diagonal matrix $D(x)$ satisfying $|x|^T (D(x) A) |x| > 0$. From (25) it also follows that $x^T (D(x) A) x > 0$ and hence $A \in P_\mathbb{R}$ by criterion (P2). Since $A \in Z_\mathbb{R}$ and $A \in P_\mathbb{R}$, we obtain that $A$ is an $M$-matrix.

Now, let $K \subseteq M_\mathbb{R}$, be a compact convex set of matrices. Using (23), one can obtain necessary and sufficient conditions for $K \subseteq M_\mathbb{R}$, similar to the nonsingularity criteria (12), (13) and (14). Note that each of these conditions is implied by condition (c) of Theorem 2.1 and the nonsingularity criteria (12), (13) and (14).

**Theorem 3.1.** Let $K \subseteq Z_\mathbb{R}$, be a compact convex set and let $\mathcal{A}$ be the set of its extreme points. The following conditions are equivalent:

(a) $K \subseteq M_\mathbb{R}$;

(b) for every nonzero $x \in \mathbb{R}^n$ with $x \geq 0$, there is some nonnegative diagonal matrix $D = D(x) \in M_\mathbb{R}$ such that

$$(26) \quad x^T (D(x) A) x > 0.$$  

(c) there is some $A_0 \in K$ which is an $M$-matrix and

$$(27) \quad K \subseteq GL_\mathbb{R}.$$  

**Proof.** (a) $\iff$ (b). Since $M_\mathbb{R} \subseteq P_\mathbb{R}$, (a) implies (b) by condition (iii) of Theorem 2.1. The converse implication follows from convexity of $K$ and (24).

(a) $\iff$ (c). Obviously, (a) $\implies$ (c). It will be shown that (c) $\implies$ (a). For any $A \in Z_\mathbb{R}$, let $\tau(A)$ be defined as $\tau(A) = \min \{\lambda | \lambda \in \sigma(A)\}$. It is well known [4] that $\tau(A) \in \sigma(A)$, i.e. $\tau(A)$ is a real eigenvalue of $A$. With the property that $\tau(A) \leq \Re e \lambda \in \sigma(A)$. Now, if $A_0 \in K$ is an $M$-matrix, we have $\tau(A_0) > 0$. Since $K$ is convex and $\tau(A)$ depends continuously on $A$, condition (27) implies that $\Re e \lambda \tau(A) > 0$, $\lambda \in \sigma(A)$ for all $A \in K$, i.e. $K \subseteq M_\mathbb{R}$.

Similarly as in Theorems 2.3, 2.4 and 2.5, one can obtain various criteria characterizing matrix sets $K, K_r, K_s$ with respect to the $M$-property of their elements. We shall state here only the simple necessary and sufficient conditions following from condition (c) of Theorem 3.1 and the nonsingularity criteria (12), (13) and (14). Note that each of $K_r, K_s$ and $K$ is a subset of $Z_\mathbb{R}$ if and only if $A_0, B \in Z_\mathbb{R}$.

Concerning (8), a criterion given in [4] states that $K \subseteq M_\mathbb{R}$ if and only if $A_0, B \in M_\mathbb{R}$ and $\sigma(BA^{-1}) \cap (-\infty, 0] = \emptyset$. Obviously, this result follows immediately from Theorem 3.1 and (12). The following corollary is also a simple consequence of this theorem and conditions (13) and (14).

**Corollary 3.2.** Let $A_0, B \in M_\mathbb{R}$. Then $K \subseteq M_\mathbb{R}$ if and only if $A_0, B \in Z_\mathbb{R}$. and $B A^{-1} \in P_\mathbb{R}$. Similarly, $K \subseteq M_\mathbb{R}$ if and only if $A_0, B \in M_\mathbb{R}$ and $B A^{-1} \in P_\mathbb{R}$. 

9 4
4. Conclusion

In this paper, we have obtained necessary and sufficient conditions characterizing the P-property and M-property of matrices belonging to a certain type of convex matrix sets. The most general results are presented in Theorems 2.1 and 3.1 which are valid for any compact convex set of matrices. The obtained criteria are formulated in terms of the extreme points of the set and generalize the well known criteria for a single P-matrix and M-matrix [4]. In comparison with the results on interval P-matrices [7], Theorem 2.1 further extends the class of matrix sets which are characterized by the P-property of their elements. Several special cases of compact convex sets of matrices are considered in Theorems 2.3, 2.4 and 2.5, respectively. These sets have been defined and studied in [5] with respect to the nonsingularity of their elements. Here, we have derived criteria which enable to establish the P-property and M-property of all elements in the corresponding set. Some of the obtained criteria are based on the results of [5] and show that there is a close relation between nonsingularity and P-property of matrices belonging to convex matrix sets.

References


Обобщение некоторых критериев для P-матриц и M-матриц

Владимир В. Монов

Институт информационных технологий, 1113 София

(Резюме)

Рассматриваются два класса специальных матриц – P-матриц и M-матриц, которые играют важную роль в области приложной линейной алгебры, интервального анализа и исследования динамического поведения больших систем управления. Дискутируются некоторые критерии, характеризующие заданную квадратную матрицу, такие как P-матрицу или M-матрицу. Основные результаты представлены в пяти теоремах.