International Journal of Pure and Applied Mathematics

Volume 49 No. 3 2008, 381-389

NEW GENERALIZED UPPER TRACE BOUND FOR THE SOLUTION OF THE LYAPUNOV EQUATION

Svetoslav Savov[1 §], Ivan Popchev[2]

Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2, Sofia, 1113, BULGARIA
[1]e-mail: savovsg@yahoo.com

Abstract: A new upper trace bound, which generalizes the best available similar estimates, improves their tightness and holds under a less conservative validity restriction imposed on the coefficient matrix, is derived for the solution matrix of the continuous algebraic Lyapunov equation (CALE). Two numerical examples illustrate the applicability of the main result.

Key Words: Hurwitz matrix, Lyapunov equation, singular value decomposition, trace

1. Introduction

The CALE plays an important role in control theory. There exist many numerical algorithms, which can be applied to compute the solution matrix, but sometimes only some estimates for it are sufficient. Estimate based approaches to study robust stability of uncertain stochastic systems and to evaluate the stability margin for real polynomials were applied in [2] and [8], respectively. The continuous algebraic Riccati equation has an wide application in optimal control design and signal processing, but since it is quadratic in the unknown matrix a direct solution may be of some difficulty, especially when the dimension is high. The available upper bounds hold for some very special cases. A result obtained in [3] states that the positive semi-definite solution of the Ric-

Received: August 14, 2008

§Correspondence author

cati equation can be bounded from above by the solution matrix of a respective CALE. These are some of the reasons, due to which the estimation problem for the CALE has attracted considerable attention for more than 30 years. All available results obtained during the first two decades were summarized in [5] and [7].

In practical applications, especially for stability/performance analysis, upper trace bounds are desired. All bounds proposed till 1997 are valid under the very restrictive assumption that the symmetric part of the coefficient matrix A is a negative definite matrix. The condition, under which it is possible to derive a valid upper trace bound, when the available similar ones fail, was investigated in [9]. By means of the singular value decomposition approach (SVD) it was shown that there exists a set of stable matrices, which is less demanding with respect to upper bounds' validity. Recently, it was proved in [10], that this "less restrictive" set can be additionally extended, which leads to even more relaxed validity constraints.

This research is entirely devoted to the other (more or less neglected) aspect of the estimation problem - tightness of upper bounds. The paper is organized as follows. The best in the sense of tightness available upper trace bounds are presented and discussed briefly in Section 2. A new upper trace bound generalizing them is derived in Section 3. It is proved that this bound is tighter and less conservative with respect to validity constraints, as well. A comparative study of the presented bounds is performed for two numerical examples.

2. Preliminaries and Previous Results

The following standard notations will be used. The symmetric part of a matrix A is $A_s = 0.5(A^T + A)$ and $A > B$ ($A \geq B$) means that $A - B$ is a positive (semi-) definite matrix. The eigenvalues of a $n \times n$ symmetric matrix S are denoted $\lambda_i(S)$, $i = 1, 2, ..., n$ and it is assumed that they are arranged in a non-ascending order, i.e., $\lambda_i(S) \geq \lambda_{i+1}(S)$, $i = 1, 2, ..., n-1$; $\mu(A) = \lambda_1(A_s)$. The $n \times n$ identity matrix and the sum of eigenvalues (trace) of a matrix M are denoted I and $tr(M)$, respectively.

Definition. It is said that L is a Lyapunov matrix (LM) for A, if $A^T L + LA < 0$ and $L > 0$.

Consider the CALE

$$A^T P + PA = -Q, \quad A,\ Q \in \mathbf{R}_{n \times n},\ Q > 0, \tag{1}$$

where A is a Hurwitz (negative stable) matrix and P denotes the unique positive definite solution matrix. The tightest upper trace bound obtained till 1997

$$t = \sum_{1}^{n} \frac{\lambda_i(Q)}{-2\lambda_i(A_s)}, \quad A_s < 0 \tag{2}$$

was derived in [4]. The above extremely conservative validity constraint was removed for the upper trace bounds proposed in [1]:

$$t_1(L) = \lambda_1(L)\frac{tr(QL^{-1})}{-2\mu(\tilde{A})}, \quad \tilde{A} = L^{\frac{1}{2}}AL^{-\frac{1}{2}}, \quad \tilde{A} < 0\,, \tag{3}$$

$$t_2(L) = \lambda_1(L)\sum_{1}^{n} \frac{\lambda_i(QL^{-1})}{-2\lambda_i(\tilde{A}_s)}, \quad \tilde{A}_s < 0\,, \tag{4}$$

where L is a LM for A. Recently, the following lower and upper matrix bounds

$$P \geq P_L(M) = S^{-1}[S(Q-M)S]^{\frac{1}{2}}S^{-1}, \quad S = (AM^{-1}A^T)^{\frac{1}{2}}; \quad Q > M > 0,$$

$$P \leq P_U(M) = \{Q + 0.5\lambda_1[-Q(A_s)^{-1}](A+I)^T(A+I) - A^TP_LA\}; \quad A_s < 0,$$

were proved in [6]. It follows, that

$$t(M) = tr[P_U(M)], \quad A_s < 0 \tag{5}$$

is an upper trace bound for P in (1). Its tightness depends on the parameter matrix M.

Consider the SVD of the coefficient matrix in (1), i.e. $A = U\Sigma V^T$, where $UU^T = VV^T = I$ and Σ is a diagonal matrix containing the singular values of A. This helps to represent A as a product of a unitary and a positive definite matrix, as follows:

$$A = FR = SF, \qquad F = UV^T, \qquad R = (A^TA)^{\frac{1}{2}}, \qquad S = (AA^T)^{\frac{1}{2}}. \tag{6}$$

Define the matrix sets $\mathbf{A} = \{A: \ A_s < 0\}$ and $\mathbf{F} = \{A: \ F_s < 0\}$. It was proved that: $A \in \mathbf{A}$ if only if R is a LM for A and

$$\mathbf{A} \subseteq \mathbf{F}; \quad A \in \mathbf{F} \text{ if and only if } R \text{ and } S^{-1} \text{ are } LMs \text{ for } A \tag{7}$$

in [9] and [10], respectively, which made possible to obtain the following upper bounds

$$P \leq P_U(L) = \eta(L)L, \quad \eta(L) = 0.5\lambda_1\{-Q[(LA)_s]^{-1}\}, L = R, S^{-1}; A \in \mathbf{F}, \tag{8}$$

$$tr(P) \leq t(L) = \min\left\{tr[P_U(L)], \frac{tr(QL^{-1})}{-2\mu(AL^{-1})}\right\}, \quad L = R, S^{-1}; \quad A \in \mathbf{F}. \tag{9}$$

The estimation problem has two important aspects-validity of the upper bounds and their tightness. Due to (7), the trace bound in (9) is less conservative with respect to validity in comparison with the bounds in (2) and (5). It was proved, that whenever the bounds in (3) and (4) hold, there exist respective valid bounds based on the SVD of the transformed matrix $\tilde{A}$ (see Theorem 3.3 in [9] and corollary 1 in [10] for details), since $\tilde{A} \in \mathbf{A}$ implies $\tilde{A} \in \mathbf{F}$. In other words, the SVD approach does not lead to conservatism with respect to validity of the derived via it bounds. This paper considers the second aspect of the estimation problem.

3. The Generalized Upper Trace Bound

Consider the CALE (1) rewritten as follows:

$$A^T(P-B)+(P-B)A=-(Q+A^TB+BA)=-Q(B), \quad B=B^T. \tag{10}$$

Theorem 3.1. *Let L be a LM for A. If P_L and P_U satisfy the matrix inequalities $0 \leq P_L \leq P \leq P_U$, then the following generalized upper trace bound for P is valid:*

$$t_g=\min\{t(L,P_L),\ t(L,P_U)\}, \tag{11}$$

$$t(L,P_L)=\frac{tr[Q(P_L)L^{-1}]}{-2\mu(AL^{-1})}+tr(P_L), \tag{12.1}$$

$$t(L,P_U)=\frac{tr[Q(P_U)L^{-1}]}{-2\rho(AL^{-1})}+tr(P_U). \tag{12.2}$$

Proof. It is based mainly on the well-known fact, that if X is a positive definite matrix, then $tr(XY) \geq \rho(X)tr(Y)$, if $Y \geq 0$ and $tr(XY) \geq \mu(X)tr(Y)$, if $Y \leq 0$. Consider the CALE (10) post-multiplied by L^{-1}, i.e., $A^T(P-B)L^{-1}+(P-B)AL^{-1}=-Q(B)L^{-1}$. Application of the tr operator to both sides of this matrix equality leads to:

$$\begin{aligned} tr[Q(B)L^{-1}] &= -tr[(L^{-1}A^T+AL^{-1})(P-B)] \\ &= 2tr[(-AL^{-1})_s(P-B)]=\gamma(L,B). \end{aligned} \tag{13}$$

If L is a LM for A, i.e., $0>\mu(AL^{-1}) \geq \rho(AL^{-1})$, for $B=P_L$ and $B=P_U$, the scalar $\gamma(L,B)$ can be bounded respectively as follows:

$$\begin{aligned} P-P_L \geq 0 \Rightarrow tr[Q(P_L)L^{-1}] &= \gamma(L,P_L) \\ &\geq 2\rho(-AL^{-1})tr(P-P_L)=-2\mu(AL^{-1})tr(P-P_L), \end{aligned}$$

$$\begin{aligned} P-P_U \leq 0 \Rightarrow tr[Q(P_U)L^{-1}] &= \gamma(L,P_U) \\ &\geq 2\mu(-AL^{-1})tr(P-P_U)=-2\rho(AL^{-1})tr(P-P_U). \end{aligned}$$

This proves the bounds in (12.1) and (12.2).

Corollary 3.1. *Consider the tracē bound in (11). For any matrices L, $P_L \neq 0$ and P_U, one has:*

$$t_g \leq t_0 = \min\{t(L,0),\ tr(P_U)\}. \tag{14}$$

In addition, if P_L is a strictly positive definite matrix, then the equality sign in (14) is possible if and only if $t_g = t_0 = tr(P)$, where, either $tr(P) = t(L,0)$, or $P = P_U \Rightarrow tr(P) = tr(P_U)$.

Proof. The trace inequality in (14) holds if:

$$t(L,P_L) \leq t(L,0),\ \forall L,\ \forall P_L \neq 0, \tag{15}$$

$$t(L,P_U) \leq tr(P_U),\ \forall L,\ \forall P_U. \tag{16}$$

Consider the bound in (12.1). The scalar $tr[Q(P_L)L^{-1}]$ is rewritten as follows:

$$\begin{aligned} tr[Q(P_L)L^{-1}] &= tr[QL^{-1} + (A^T P_L + +P_L A)L^{-1}] \\ &= tr[QL^{-1} + (L^{-1}A^T + AL^{-1})P_L] = tr[QL^{-1} + 2(AL^{-1})_s P_L] \end{aligned}$$

and its substitution in (12.1) results in the following trace inequality:

$$t(L,P_L) = \frac{tr\{QL^{-1} + 2[(AL^{-1})_s - \mu(AL^{-1})I]P_L\}}{-2\mu(AL^{-1})} \leq \frac{tr(QL^{-1})}{-2\mu(AL^{-1})} = t(L,0),$$

which holds for any matrix $P_L \geq 0$, since $(AL^{-1})_s - \mu(AL^{-1})I$ is a negative semi-definite matrix for any L. This proves the trace inequality in (14). Let $t(L,P_L) = t(L,0)$, $P_L > 0$. It follows that $tr\{[(AL^{-1})_s - \mu(AL^{-1})I]P_L\} = 0$, which is possible if and only if $(AL^{-1})_s = \mu(AL^{-1})I$. In this case, for $B = P_L$ the trace equality in (13) becomes

$$\begin{aligned} tr[Q(P_L)L^{-1}] = tr[QL^{-1} + 2\mu(AL^{-1})P_L] &= -2\mu(AL^{-1})tr(P - P_L) \\ &\Rightarrow \frac{tr(QL^{-1})}{-2\mu(AL^{-1})} = t(L,0) = tr(P). \end{aligned}$$

Consider the bound in (12.2). Then, (15) holds if $tr[Q(P_L)L^{-1}]$ is a non-positive scalar for arbitrary matrices P_U and L. Having in mind (10), for $B = P_U$, the following inequality holds:

$$\begin{aligned} tr[Q(P_U)L^{-1}] &= tr\{L^{-\frac{1}{2}}[A^T(P_U - P) + (P_U - P)A]L^{-\frac{1}{2}}\} \\ &= 2tr[\tilde{A}_s L^{-\frac{1}{2}}(P_U - P)L^{-\frac{1}{2}}] \leq 0,\quad \forall P_U,\quad \forall L, \end{aligned}$$

since $\tilde{A} \in \mathbf{A}$ and $P \leq P_U$; the equality sign is possible if and only if P_U coincides with the solution matrix. This proves the inequality in (16) and completes the proof of the corollary.

Consider the bounds $t(L)$ in (3) and $t(L,P_L)$ in (12.1). The matrix measure of $\tilde{A}$ can be evaluated as follows:

$$0 > 2\mu(\tilde{A}) = \lambda_1[(L^{-1}A^T + AL^{-1})L] \leq \lambda_1(L^{-1}A^T + AL^{-1})\lambda_1(L) = 2\mu(AL^{-1})\lambda_1(L).$$

Therefore, $t(L) \geq t(L,0) \geq t(L,P_L)$, $\forall L$, $\forall P_L \geq 0$, i.e., t_g is tighter than $t(L)$ and all upper trace bounds of the type $tr(P_U)$, $P \leq P_U$, (e.g. the bound in (5)), in accordance with corollary 3.1. The next lemma provides a bound, which generalizes the trace estimate in (4).

Lemma 3.1. *Let L be a LM for A. If $Q(P_L)$ in (10) is a positive semi-definite matrix, then*

$$tr(P) \leq \tilde{t}(L,P_L) = \lambda_1(L)\sum_1^n \frac{\lambda_i[Q(P_L)L^{-1}]}{-2\lambda_i(\tilde{A}_s)} + tr(P_L), \quad \tilde{A} = L^{\frac{1}{2}}AL^{-\frac{1}{2}}. \quad (17)$$

Proof. Pre- and post-multiplication of (10) by $L^{-\frac{1}{2}}$ results in the following modified CALE:

$$\tilde{A}^T[L^{-\frac{1}{2}}(P-P_L)L^{-\frac{1}{2}}] + [L^{-\frac{1}{2}}(P-P_L)L^{-\frac{1}{2}}]\tilde{A} = -L^{-\frac{1}{2}}Q(P_L)L^{-\frac{1}{2}}.$$

If $Q(P_L) \geq 0$, then $L^{-\frac{1}{2}}(P-P_L)L^{-\frac{1}{2}} \geq 0$, and application of the upper trace bound (2) to the solution matrix of the above CALE results in the following trace inequalities:

$$\frac{1}{\lambda_1(L)}tr(P-P_L) \leq tr[L^{-\frac{1}{2}}(P-P_L)L^{-\frac{1}{2}}] \leq \sum_1^n \frac{\lambda_i[Q(P_L)L^{-1}]}{-2\lambda_i(\tilde{A}_s)} \Rightarrow tr(P) \leq \tilde{t}(L,P_L).$$

One can easily verify, that the estimates in (2) and (4) are particular cases of this bound, since they are obtained from (17) for $L = I$, $P_L = 0$ and $P_U = 0$, respectively. The results stated by Theorem 3.1 and Lemma 3.1 are based on the assumption that the matrices L, P_L and P_U are known, which is always possible in theory. If a LM L for A has been computed (e.g., by the solution of a linear matrix inequality), then there always exist some appropriate positive scalars α and β, such that for $P_L = \alpha L$ and $P_U = \beta L$ one has $Q(P_L) \geq 0$ and $Q(P_U) \leq 0$ in (10). This implies the matrix inequalities $0 < P_L \leq P \leq P_U$, in accordance with Lyapunov's Stability Theorem. Unfortunately, the determination of a LM, such that the tightest bound is obtained is a difficult and open problem (the same refers to the parameter matrix M in (5)). This may result in the practical inapplicability of such bounds. That is the reason why the solution of the estimation problem for the CALE has been searched for by imposing restrictions on matrix A for validity of the upper bounds. The definition of the set helped to derive the bound (9), which holds under relaxed validity constraints and the application of Theorem 3.1 makes possible its improvement. Consider the positive definite matrix

$$P_L(L) = \theta(L)L, \quad \theta(L) = 0.5\lambda_n\{-Q[(LA)_s]^{-1}\}, \; L = R, \; S^{-1}, \qquad (18)$$

where the matrices R and S are given in (6). For $B = P_L(L)$, $Q[P_L(L)]$ in (10) is a positive semi-definite matrix, which guarantees that $P_L(L)$ is a valid lower matrix bound for P in (1).

Lemma 3.2. *If $A \in \mathbf{F}$, then for $L = R$, S^{-1} in (6), $P_L = P_L(L)_{(18)}$ and $P_U = P_U(L)_{(8)}$, one has*

$$tr(P) \leq \tilde{t}_g = \min\{t_{g_{(11)}}, \tilde{t}(L, P_L)_{(17)}\}. \qquad (19)$$

Proof. If $A \in \mathbf{F}$, then R and S^{-1} are LMs for A in accordance with (7). Matrices P_L in (18) and P_U in (8) represent a lower and upper matrix bound for P, respectively. Therefore, these matrices meet all assumptions made in Theorem 3.1 and Lemma 3.1, which proves the assertion of the lemma.

According to corollary 3.1, the generalized bound in (19) is a tighter upper trace estimate for P than $t(L)$ in (9) and it holds under the same relaxed validity constraint. Note, that $\tilde{t}_g$ is comprised of 12 different upper trace bounds.

4. Examples

Example 1. (see [1]) Let $\Delta[\%] = (\frac{tr \text{ bound}}{\text{solution } tr} - 1)X\,100$ be a upper trace bound error indicator and

$$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \in \mathbf{A}, \quad Q = I \Rightarrow P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

Bounds (2) and (5) do not hold . The LM for A has been chosen in [1] to be a diagonal matrix, given by $L = diag[0.25, \; 1]$. The bounds in (3) and (4) are: $t_1(L) = 5 > t_2(L) = 13/3$ (the authors computed wrongly the bound in (4) as $7/3$). Since $F_s = -0.7071I$, i.e., $\tilde{A} \in \mathbf{F}$, the SVD approach is directly applicable, which leads to the following interesting results. The lower and upper matrix bounds for P in (18) and (8), computed for $L = S^{-1}$, coincide with the solution matrix P. This means that $t(S^{-1})_{(9)} = \tilde{t}_{g_{(19)}} = tr(P) = 2$. The error indicators for the valid bounds are: $\Delta_{(3)} = 150\% > \Delta_{(4)} = 117\% > \Delta_{(9)} = \Delta_{(19)} = 0\%$. This example illustrates perfectly the stated difficulties concerning the right choice for L, even in such simple cases.

Example 2. (see [6]) Let

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \mathbf{A}, \qquad Q = \begin{bmatrix} 5.0 & 0.0 & 1.0 \\ 0.0 & 8.0 & 1.4 \\ 1.0 & 1.4 & 5.4 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 2.50 & 1.25 & 0.50 \\ 1.25 & 5.25 & 0.95 \\ 0.50 & 0.95 & 2.70 \end{bmatrix}.$$

Since $A \in \mathbf{A}$, all presented bounds are valid. The following results have been obtained: $t_1(I)_{(3)} = 18.4 > t(M)_{(5)} = 17.3047 > t_{(2)} = t(I)_{(4)} = 12.92012$. For $P_L = P_L(S^{-1})$ in (18), the upper trace bound in (19) is $\tilde{t}_{g_{(19)}} = t[R, P_L(S^{-1})]_{(12.1)} = 10.5232$, which is the tightest one in this case. For a comparison, $t(R)_{(9)} = t(R,0)_{(12.1)} = 10.7687$ and the solution trace is $tr(P) = 10.45$. This example illustrates the improvement in the upper trace estimates due to Theorem 3.1 and Lemma 3.2. Note also, that the bound in (9) is tighter than all bounds based on the assumption for validity that $A \in \mathbf{A}$. The respective error indicators are computed as follows: $\Delta_{(3)} = 76\% > \Delta_{(5)} = 65.6\% > \Delta_{(2)} = 23.64\% > \Delta_{(9)} = 3.05\% > \Delta_{(19)} = 0.7\%$.

Acknowledgements

This work is supported by the Bulgarian Academy of Sciences under grant No 010077/2007.

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