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PERTURBATION BOUNDS FOR THE MATRIX EQUATION $X^s \pm A^{\mathrm{H}} X^t A = Q$

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Abstract

In this paper we consider the sensitivity of the nonlinear complex matrix equation $X^s \pm A^{\mathrm{H}} X^t A = Q$, where the exponents s and t are real numbers. Local and nonlocal perturbation bounds for the perturbation in the solution X are obtained.

Key words: perturbation analysis, nonlinear matrix equations, perturbation bounds, Lyapunov majorants

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1. Introduction and notations. We consider the nonlinear matrix equation

$$X^s \pm A^{\mathrm{H}} X^t A = Q, \tag{1}$$

where the data matrices A, Q and the solution X are $n \times n$ complex matrices, Q is positive definite and s, t are real numbers. Equations of this type arise from the areas of ladder networks, dynamic programming, control theory, stochastic filtering and statistics, etc.

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Several authors have proposed perturbation bounds for the solution of particular classes of equation (1) with $s = 1$. The case $t = -1$ is considered in [1], $t = 1$ in [19], $t = -q$, $q \in (0, 1]$ in [2], $t = -1/2$ in [3], $t = \pm 1/p$, $p = 1, 2, \ldots$ in [4], $t = -2$ in [5–7], $t = -r$, $r = 1, 2, \ldots$ in [8,9]. In [20] perturbation bounds to the matrix equation $C + \sum_{i=1}^{r} A_i X B_i + D X^s E = 0$ with r and $s \geq 2$ – positive integers are proposed.

The sensitivity of the solution of the real version of equation (1), when s is a positive integer, Q is the identity matrix and t is negative integer is studied firstly in [10]. Then in [11] the sensitivity of equation (1) with $Q > 0$ and s, t – both nonnegative integers, is considered. Perturbation estimates of the positive definite solution to the complex matrix equation (1) with s-positive integer and t-negative integer are proposed in [12,13].

In this paper we give perturbation bounds for the solution of the complex matrix equation (1) in the most general case, where s and t are real numbers. Using the technique of Frechét derivatives and applying the method of Lyapunov majorants and the Schauder fixed point principle, we obtain local and non-local perturbation bounds for the positive definite solution of equation (1). For the computation of the p-th root $X^{1/p}$ in this case the algorithms considered in [2,14,15] my be used.

Throughout the paper we use the notations $\mathbb{N}$ – the set of natural numbers; $\mathbb{R}$ – the set of real numbers; $\mathbb{C}$ – the set of complex numbers; $\mathbb{S}^{n \times n}$ – the set of all $n \times n$ positive definite matrices; I_n – the identity $n \times n$ matrix; $A^\top$, $\overline{A}$, A^{H} stands for the transpose, complex conjugate and complex conjugate transpose of A; $\mathrm{vec}(A) = \begin{bmatrix} a_1^\top, & a_2^\top, & \ldots, & a_n^\top \end{bmatrix}^\top \in \mathbb{C}^{n^2}$ – the column-wise vector representation of the matrix $A = \begin{bmatrix} a_1, & a_2, & \ldots, & a_n \end{bmatrix} \in \mathbb{C}^{n \times n}$, $a_j \in \mathbb{C}^n$, where $\mathbb{C}^n = \mathbb{C}^{n \times 1}$; $\mathrm{Mat}(\mathcal{L}) \in \mathbb{C}^{n^2 \times n^2}$ – the matrix representation of the linear matrix operator $\mathcal{L} : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$; $A \otimes B = [A(k,l)B]$ – the Kronecker product of the matrices $A = [A(k,l)]$ and B; $\| \cdot \|$ – a vector or a matrix norm; $\| \cdot \|_2$ – the Euclidean vector or the spectral matrix norm; $\| \cdot \|_{\mathrm{F}}$ – the Frobenius norm; The notation ":=" stands for "equal by definition".

2. Existence of a solution. In this section we give a theorem on the existence of a positive definite solution of equation (1).

Theorem 1. *Denote $Y := X^s$ and $\mathcal{G}(X) := \pm X^{t/s} : \mathbb{S}^{n \times n} \mapsto \mathbb{S}^{n \times n}$. If $\mathcal{G}(Y)$ is continuous and monotone, (i.e. $X \leq Y$ implies $\mathcal{G}(X) \leq \mathcal{G}(Y)$) and if $A^{\mathrm{H}} \mathcal{G}(Q) A \leq Q$, then equation (1) has a positive definite solution. If $\mathcal{G}(Y)$ is continuous and anti-monotone, (i.e. $X \leq Y$ implies $\mathcal{G}(X) \geq \mathcal{G}(Y)$) and if there exists Y_0 such that $Q - A^{\mathrm{H}} \mathcal{G}(Y_0) A \leq Y_0$, then equation (1) has a positive definite solution.*

Proof. Using the notations of $Y := X^s$ and $G(X) := \pm X^{t/s} : \mathbb{S}^{n \times n} \mapsto \mathbb{S}^{n \times n}$, we rewrite equation (1) in the equivalent form

$$Y + A^{\mathrm{H}}\mathcal{G}(Y)A = 0. \tag{2}$$

Let $\mathcal{G}(Y) : \mathbb{S}^{n\times n} \mapsto \mathbb{S}^{n\times n}$ be continuous and monotone. Assume that $A^{\mathrm{H}}\mathcal{G}(Q)A \leq Q$. Then according to Corollary 2.2 from [16] equation (2) has a solution in $[0, Q]$.

Let $\mathcal{G}(Y) : \mathbb{S}^{n\times n} \mapsto \mathbb{S}^{n\times n}$ be continuous and anti-monotone. If there exists Y_0 such that $Q - A^{\mathrm{H}}\mathcal{G}(Y_0)A \leq Y_0$, then according to Corollary 2.2 from [16] equation (2) has a positive definite solution in $[Q, Y_0]$.

Thereby under the assumptions, equation (1) has a positive definite solution. □

3. Perturbation bound. In this section we develop perturbation bounds for the solution of equation (1). We consider the perturbed matrix equation

$$\begin{aligned} F(X+\delta X, A+\delta A, Q+\delta Q) &:= (X+\delta X)^s \\ &\pm (A+\delta A)^{\mathrm{H}}(X+\delta X)^t(A+\delta A) - Q - \delta Q = 0, \end{aligned} \tag{3}$$

where δA and δQ are perturbations in the data matrices A and Q of equation (1). We assume that the perturbations are sufficiently small so that the perturbed equation (3) has a solution $X + \delta X$ in the neighbourhood of the unperturbed solution X.

3.1. Local bounds. Rewrite the perturbed equation (3) in terms of Fréchet derivatives.

$$\begin{aligned} &F(X+\delta X, A+\delta A, Q+\delta Q) \\ &\quad = F(X,A,Q) + F_X(X,A,Q)(\delta X) + F_A(X,A,Q)(\delta A) \\ &\quad + F_{\bar{A}}(X,A,Q)(\delta\bar{A}) + F_Q(X,A,Q)(\delta Q) \\ &\quad + G(X,A,Q)(\delta X, \delta A, \delta Q) = 0. \end{aligned} \tag{4}$$

Here $F(X,A,Q) := X^s \pm A^{\mathrm{H}}X^tA - Q = 0$ denotes the unperturbed terms, $G(X,A,Q)(\delta X, \delta A, \delta Q)$ contains second and higher order terms in $\delta X, \delta A, \delta Q$, $F_X(X,A,Q)(Z) := \mathcal{F}(s,X)(Z) \pm A^{\mathrm{H}}\mathcal{F}(t,X)(Z)A$ and $F_Q(X,A,Q)(Z) := -Z$ are the partial Fréchet derivatives of (3) in X and Q, respectively, computed at the point (X,A,Q). The terms $F_A(X,A,Q)(Z) := \pm A^{\mathrm{H}}X^tZ$ and $F_{\bar{A}}(X,A,Q)(Z) := \pm Z^{\mathrm{H}}X^tA$ are the partial Fréchet pseudo derivatives of (3) in A. The notation $\mathcal{F}(p,X) : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is for the Fréchet derivative of the function $X \to X^p$, $p \in \mathbb{R}$ at the point $X \in \mathbb{S}(n)$ and a given increment $E \in \mathbb{C}^{n\times n}$, with $\|E\|_2 < \lambda_{\min}(X)$

$$(X+E)^p = X^p + \mathcal{F}(p,X)(E) + \mathrm{O}(\|E\|_2),\ E \to 0.$$

Recall that for different value of $p \in \mathbb{R}$ the derivatives $\mathcal{F}(p,X)(\delta X)$ are given in [4].

Having in mind that $F(X,A,Q) = 0$ we obtain

$$\begin{aligned} F_X(X,A,Q)(\delta X) &= -F_A(X,A,Q)(\delta A) \\ &\quad -F_{\bar{A}}(X,A,Q)(\delta \bar{A}) - F_Q(X,A,Q)(\delta Q) \\ &\quad -G(X,A,Q)(\delta X, \delta A, \delta Q). \end{aligned}$$

Denote $L := \mathrm{Mat}\{F_X(X,A,Q)\} = L_s \pm (A^\top \otimes A^{\mathrm{H}})L_t$, $L_Q := \mathrm{Mat}\{F_Q(X,A,Q)\} = -I_n$, $L_A := \mathrm{Mat}\{F_A(X,A,Q)\} = I \otimes A^{\mathrm{H}} X^t$, $L_{\bar{A}} := \mathrm{Mat}\{F_{\overline{A}}(X,A,Q)\} = ((X^t A^{\mathrm{H}})^\top \otimes I)\mathcal{P}_{n^2}$. Here $\mathcal{P}_{n^2} \in \mathbb{R}^{n^2 \times n^2}$ is the so-called vec-permutation matrix, such that $\mathrm{vec}(Y^\top) = \mathcal{P}_{n^2}\mathrm{vec}(Y)$ for each $Y \in \mathbb{C}^{n \times n}$, L_s and L_t are the matrices of the corresponding Fréchet derivatives $\mathcal{F}(s,X)$ and $\mathcal{F}(t,X)$ for $p = s,t \in \mathbb{R}$. Expressions of L_p for different values of $p \in \mathbb{R}$ are given in [4]. Supposing that the operator $F_X(X,A,Q)$ is invertible, i.e its matrix L is non-singular, one obtains

$$\delta X = -F_X^{-1} \circ F_Q(\delta Q) - F_X^{-1} \circ F_A(\delta A) - F_X^{-1} \circ F_{\bar{A}}(\delta \bar{A}) + \mathrm{O}(\|\delta\|^2),\ \ \delta \to 0,$$

or in a vector form after applying the vec operator

$$\mathrm{vec}(\delta X) = W_Q \mathrm{vec}(\delta Q) + (W_A + W_{\bar{A}})\mathrm{vec}(\delta A) + \mathrm{O}(\|\delta X\|^2). \tag{5}$$

Here $W_Q := -L^{-1}L_Q$, $W_A = -L^{-1}L_A$, $W_{\bar{A}} = -L^{-1}L_{\bar{A}}$.

Denote $\delta_X := \|\delta X\|_{\mathrm{F}}$, $\delta := \left[\begin{array}{cc} \delta_1 & \delta_2 \end{array}\right]^\top = \left[\begin{array}{cc} \|\delta A\|_{\mathrm{F}} & \|\delta Q\|_{\mathrm{F}} \end{array}\right]^\top \in \mathbb{R}^2_+$. Relation (5) gives

$$\delta_X \leq \| \left[\begin{array}{cc} W_Q^{\mathcal{R}} & M_A \end{array}\right] \|_2 \|\delta\|_2 + \mathrm{O}((\delta_X + \delta)^2), \delta_x, +\delta \to 0,$$

$$\delta_X \leq \sqrt{\delta^\top R \delta} + \mathrm{O}((\delta_X + \delta)^2), \delta_x, +\delta \to 0,$$

where $W_Q^{\mathcal{R}} := \left[\begin{array}{cc} W_{Q0} & -W_{Q1} \\ W_{Q1} & W_{Q0} \end{array}\right] \in \mathbb{R}^{2n \times 2n}$ is the real representation of the matrix

$$W_Q = W_{Q0} + iW_{Q1} \in \mathbb{C}^{n \times n},$$

$$M_A := M(W_A, W_{\bar{A}}) = \left[\begin{array}{cc} W_{A0} + W_{\bar{A}0} & W_{\bar{A}1} - W_{A1} \\ W_{A1} + W_{\bar{A}1} & W_{A0} - W_{\bar{A}0} \end{array}\right] \in \mathbb{R}^{2n \times 2n}$$

and

$$W_A = W_{A0} + iW_{A1}, \quad W_{\bar{A}} = W_{\bar{A}0} + iW_{\bar{A}1} \in \mathbb{C}^{n\times n}$$

with $A_{A0}, A_{A1}, A_{\bar{A}0}, A_{\bar{A}1} \in \mathbb{R}^{n\times n}$. The matrix R is given by

$$R = \begin{bmatrix} \|(W_Q^{\mathcal{R}})^\top W_Q^{\mathcal{R}}\|_2 & \|(W_Q^{\mathcal{R}})^\top M_A\|_2 \\ \|M_A^\top W_Q^{\mathcal{R}}\|_2 & \|M_A^\top M_A\|_2 \end{bmatrix} \in \mathcal{R}_+^2.$$

Hence, we formulate the following local perturbation bound for the perturbation δX in the solution X of equation (1)

$$(6) \quad \delta_X \leq \mathrm{est}(\delta)$$
$$:= \min\{\| \begin{bmatrix} W_Q^{\mathcal{R}} & M_A \end{bmatrix} \|_2 \|\delta\|_2, \sqrt{\delta^\top R\delta}\} + \mathrm{O}((\delta_X + \delta)^2), \delta_x, +\delta \to 0,$$

3.2. Non-local bound. In this section we present a nonlocal perturbation bound within second order terms in δX for the solution of equation (1). We rewrite the perturbed equation (3) in the form

$$F_X(\delta X) = -\Phi_0(\delta A, \delta Q) - \Phi_1(\delta X, \delta A, \delta Q) - \Phi_2(\delta X, \delta A, \delta Q) + \mathrm{O}(\|\delta X\|^3),$$

with

$$\begin{aligned} \Phi_0(\delta A, \delta Q) &:= F_Q(X, A, Q)(\delta Q) + F_A(X, A, Q)(\delta A) + F_{\bar{A}}(X, A, Q)(\delta \bar{A}) \\ &\quad +\delta A^{\mathrm{H}} X^t \delta A, \\ \Phi_1(\delta X, \delta A, \delta Q) &:= \pm(A^{\mathrm{H}}\mathcal{F}(t, X)(\delta X)\delta A \\ &\quad \pm \delta A^{\mathrm{H}}\mathcal{F}(t, X)(\delta X)A \pm \delta A^{\mathrm{H}}\mathcal{F}(t, X)(\delta X)\delta A), \\ \Phi_2(\delta X, \delta A, \delta Q) &:= \pm(A + \delta A)^{\mathrm{H}}\mathcal{F}_2(t, X)(\delta X)(A + \delta A), \end{aligned}$$

where the expression $\mathcal{F}_2(t, X)(\delta X) := (X+\delta X)^t - X^t - \mathcal{F}(t, X)(\delta X) = \mathrm{O}(\|\delta X\|^2)$ contains the terms of second and higher order in δX.

Assuming the invertibility of the operator $F_X(X, A, Q)$ we obtain the following vector operator equation within the terms of second order

$$\begin{aligned} \mathrm{vec}(\delta X) &= \Theta(\delta X, \delta A, \delta Q), \\ \Theta(\delta X, \delta A, \delta Q) &:= -L^{-1}\mathrm{vec}(\Phi_0(\delta A, \delta Q)) - L^{-1}\mathrm{vec}(\Phi_1(\delta X, \delta A, \delta Q)) \\ &\quad - L^{-1}\mathrm{vec}(\Phi_2(\delta X, \delta A, \delta Q)). \end{aligned}$$

Let $\delta_X \leq \rho$, where ρ is some positive quantity. The Frobenius norm of $\Theta(\delta X, \delta A, \delta Q)$ may be estimated as

$$\|\Theta(\delta X, \delta A, \delta Q)\|_{\mathrm{F}} \leq a_2(\delta)\rho^2 + a_1(\delta)\rho + a_0(\delta),$$

where

$$\begin{aligned} a_0(\delta) &:= \mathrm{est}(\delta) + \|L^{-1}\|_2 \|X^t\|_2 \delta_A^2, \\ a_1(\delta) &:= \|L^{-1}\|_2 \|L_t\|_2 \left(2\|A\|_2 \delta_A + \delta_A^2\right), \\ a_2(\delta) &:= \|L^{-1}\|_2 (\|A\|_2 + \delta_A)^2 \varphi(t, X). \end{aligned} \tag{7}$$

Here $\mathrm{est}(\delta)$ is the local bound (6), L is the matrix representation of the operator $F_X(X, A, Q)$ and $\varphi(t, X)$ denotes $\|\mathrm{Mat}(\mathcal{F}_2(t, X))\|_2 \leq \varphi(t, X)\delta_X^2 + \mathrm{O}(\|\delta X\|^3)$.

The Lyapunov majorant [17,18] for the vector operator equation $\mathrm{vec}(\delta X) = \Theta(\delta X, \delta A, \delta Q)$ such that $\|\Theta(\delta X, \delta A, \delta Q)\|_{\mathrm{F}} \leq h(\rho, \delta)$ is a quadratic function $h(\rho, \delta) = a_2(\delta)\rho^2 + a_1(\delta)\rho + a_0(\delta)$.

Suppose that $\delta \in \Omega$, where

$$\Omega := \left\{\delta \geq 0 : a_1(\delta) + 2\sqrt{a_0(\delta)a_2(\delta)} \leq 1\right\} \subset \mathbb{R}_+^{k+1}. \tag{8}$$

The inclusion $\delta \in \Omega$ guarantees that the equation $h(\rho, \delta) = \rho$ has a root

$$\rho = f(\delta) := \frac{2a_0(\delta)}{1 - a_1(\delta) + \sqrt{(1 - a_1(\delta))^2 - 4a_0(\delta)a_2(\delta)}}. \tag{9}$$

Hence, the operator $\Theta(\cdot, \delta A, \delta Q)$ maps the closed convex ball $\mathcal{B}(\delta) := \{H \in \mathbb{C}^{n\times n} : \|H\|_{\mathrm{F}} \leq f(\delta), \delta \in \Omega\} \subset \mathbb{C}^{n\times n}$ into itself. According to the Schauder fixed point principle there exists a solution $\delta X \in \mathcal{B}(\delta)$ of equation $h(\rho, \delta) = \rho$, or, equivalently, $a_2(\delta)\rho^2 - (1 - a_1(\delta))\rho + a_0(\delta) = 0$, for which $\delta_X = \|\delta X\|_{\mathrm{F}} \leq f(\delta), \delta \in \Omega$. Hence, we formulate a non-local perturbation bound for the solution of equation (1).

Theorem 2. *Let $\delta \in \Omega$, where Ω is given in* (8). *Then within second order terms in δX the non-local perturbation bound $\delta_X \leq f(\delta)$ is valid for equation* (1), *where $f(\delta)$ is determined by relations* (9) *and* (7).

The expressions of L_t and bounds for $\varphi(t, X)$ for different values of $t \in \mathbb{R}$: $t = \pm r$, $t = \pm 1/2$, $t = 1/3$, $t = \pm 1/p$, $t = \pm r/p$, $r, p = 1, 2, \ldots,$ are listed in Table 1.

Table 1

t	L_t	$\varphi(t,X)$
r	$\sum_{k=0}^{r-1}(X^k)^\top \otimes X^{r-1-k}$	$\sum_{j=2}^{r}\left\|\sum_{i=j}^{r}\left(X^{r-i}\right)^\top \otimes X^{i-j}\right\|_2 \left\|X^{j-2}\right\|_2 \le \left(r-1+\binom{r}{2}\right)\|X\|_2^{(r-2)}$
$-r$	$-\sum_{k=0}^{r-1}(X^{k-r})^\top \otimes X^{-(1+k)}$	$\sum_{j=1}^{r}\left\|\sum_{i=1}^{r-j+1}(X^{-r+i+j-2})^\top \otimes X^{-i}\right\|_2 \left\|X^{-j}\right\|_2 \le \left(2r-1+\binom{r}{2}\right)\left\|X^{-1}\right\|_2^{(r+2)}$
$\frac{1}{2}$	$I\otimes X + X^\top \otimes I$	$2\|L_{\frac{1}{2}}\|_2^3/(1-2\|L_{\frac{1}{2}}\|_2^2\delta_X+\sqrt{1-4\|L_{\frac{1}{2}}\|_2^2\delta_X})$
$-\frac{1}{2}$	$\left(-(X^{-1})^\top \otimes X^{-1/2}-(X^{-1/2})^\top \otimes X^{-1}\right)^{-1}$	$\frac{3}{8}\|(X^{-1})^\top \otimes X^{-1/2}\|_2\|X^{-1}\|_2,$
$\frac{1}{3}$	$I\otimes X^2 + X^\top \otimes X + (X^2)^\top \otimes I$	$3\|X\|_2^{1/3}\|L_{1/3}\|_2^3$
$\frac{1}{p}$	$\left(\sum_{k=0}^{p-1}(X^{k/p})^\top \otimes X^{(p-1-k)/p}\right)^{-1}$	$(\|X\|_2^{(p-2)/p}\|L_{1/p}\|_2^3 p(p-1))/2$
$-\frac{1}{p}$	$\left(-\sum_{k=0}^{p-1}(X^{(k-p)/p})^\top \otimes X^{-(1+k)/p}\right)^{-1}$	$--$
$\frac{r}{p}$	$\left(\sum_{k=0}^{r-1}(X^{k/p})^\top \otimes X^{(r-1-k)/p}\right)L_{1/p}$	$--$
$-\frac{r}{p}$	$-\left((X^{-r/p})^\top \otimes X^{-r/p}\right)L_{r/p}$	$--$

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