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RELAXED ROBUST STABILITY ANALYSIS

Svetoslav Savov, Ivan Popchev

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Abstract

Robust stability of polytopic systems is analysed via affine Lyapunov function (LF). It is shown that when the pairwise inequalities between the entries of the uncertain vector are taken into account less conservative and relaxed conditions are obtained.

Key words: affine Lyapunov function, matrix polytope, LMI, robust stability

1. Introduction. Stability analysis of linear systems subjected to structured real parametric uncertainty belonging to a compact vector set has been recognized as a key issue in the analysis of control systems. Robust stability cannot be directly assessed using convex optimization. In order to reduce the gap between quadratic and robust stability, attempts for reducing the conservatism of LMI methods have been made for more than a decade. Aimed at going beyond parameter-independent LFs, LMI techniques were proposed to derive quadratic in the state candidates for Lyapunov functions, which are affine [5, 6, 10], quadratic [1] and recently polynomial [2–4, 9], in the uncertain parameter. Robust stability is verified through convex optimization problems formulated in terms of parameterized LMIs, which can be efficiently solved by polynomial-time algorithms.

The objective of this research is to find computable, less conservative and relaxed robust stability conditions via affine LFs, in a case when the uncertain vector α belongs to the unit simplex. It is actually motivated by several recently obtained results [5, 6, 10], aimed at solving the same problem, which exhibit some common shortcomings (sources of conservatism). The main contributions are: (i)

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a new necessary and sufficient condition for validity of affine LFs based on the theory of M-matrices is obtained, (ii) aimed at taking some additional advantage from the fact that α is a nonnegative vector some or all pairwise inequalities between its entries are also considered, which results in a new alternative necessary and sufficient robust stability condition, and (iii) three generalizing robust stability conditions proved to be less conservative in comparison with the available ones are obtained.

2. Notations, previous results and open problems. The notations $A>0$ ($A \geq 0$) and $a > 0$ indicate that A is a positive (semi-positive) definite matrix and a is a positive vector; $A = [a_{ij}] \in \mathbb{R}_n$ and $a = (a_i) \in \mathbb{R}^N$ denote real $n \times n$ matrix and $N \times 1$ vector with entries a_{ij} and a_i respectively. The sum of N nonnegative scalars α_i is $|\alpha|$, $\lambda_i(A)$ is the i-th eigenvalue of a matrix A and $x_n \equiv \{x \in \mathbb{R}^n : x^T x = 1\}$. Consider the uncertain linear system

$$(1) \qquad \dot{x} = A(\alpha)x, \quad A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i \in \mathbb{R}_n, \quad \alpha \in \omega_N \equiv \{\alpha \in \mathbb{R}^N : |\alpha| = 1\},$$

where all vertices A_i are Hurwitz (negative stable) matrices and the associated with it candidate for a LF $v(\alpha, x) = x^T P(\alpha)x$, $x \in x_n$, $P(\alpha) = \sum\limits_{i=1}^{N} \alpha_i P_i$, $P_i > 0$. It is desired to find conditions under which the time derivative of $v(\alpha, x)$ is negative on ω_N and x_n, i.e.,

$$(2) \;\; 0 < -\dot{v}(\alpha, x) = -x^T[A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha)]x = x^T \Pi(\alpha, 2)x = 0.5\alpha^T C(x)\alpha,$$

for all α, x, where $\Pi(\alpha, 2) = \sum\limits_{\substack{i,j=1\\ i\leq j}}^{N} \alpha_i \alpha_j \Pi_{ij}$ is a HMP in α of degree 2 with $0.5N(N+1)$ coefficients $\Pi_{ij} = -\pi_{ij}(A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j)$, $\pi_{ij} = 0.5$, $i = j$, and $\pi_{ij} = 1$, $i < j$. The $N \times N$ symmetric matrix $C(x) = [c_{ij}(x)] = [\pi_{ij}^{-1} x^T \Pi_{ij} x]$, $i \leq j$, is said to be a coefficient matrix (CM). Consider a HMP $\tilde{\Pi}(\alpha, 2) = \sum\limits_{\substack{i,j=1\\ i\leq j}}^{N} \alpha_i \alpha_j V_{ij}$, $V_{ij} \leq \Pi_{ij}$, $i \leq j$, i.e., $-\dot{v}(\alpha, x) \geq x^T \tilde{\Pi}(\alpha, 2)x = 0.5\alpha^T \tilde{C}(x)\alpha$, for all α, x. The inequality (2) holds if there exist matrices V_{ij}, such that the following conditions are valid.

Theorem 2.1 ([5]). $\sum\limits_{\substack{j=1\\ j\geq i}}^{N} \pi_{ij}^{-1} V_{ij} > 0$, $i = 1, \ldots, N$, $V_{ij} \leq 0$, $i < j$.

Theorem 2.2 ([5,6]). *For $V_{ij} = c_{ij} I$, $i \leq j$, the matrix $\tilde{C}(x) = C = [c_{ij}]$ is positive definite.*

Unfortunately, these results are rather conservative. In order to apply the Frobenius-Perron theorem and the theory of M-matrices, the validity of (2) is analysed in Theorem 2.1 via some matrix $\tilde{C}(x)$, with artificially forced to be non-positive on x_n off-diagonal entries. The N LMIs, equivalent to the vector inequality $\tilde{C}(x)\beta_1 > 0$, for all x, $\beta_1 = (1)^T \in \mathbb{R}^N$, guarantee sufficiently that $\tilde{C}(x)$ is a positive definite matrix, but one fixed vector β_1 is required for all x. In Theorem 2.2, in order to remove the awkward dependence of the CM $C(x)$ on vector x, the coefficients of $\Pi(\alpha, 2)$ are bounded from below by some matrices $c_{ij}I$, which is a very conservative estimation. It will be shown that there exists an efficient way to eliminate (to a great extent) the shortcomings demonstrated by Theorems 2.1 and 2.2, which leads to guaranteed less conservative and relaxed tests for robust stability of system (1) using function $v(\alpha, x)$.

3. Main results. Let $\boldsymbol{L}$ denote the set of real $N \times N$ matrices with non-positive off-diagonal entries. The set of M-matrices consists of all $M \in \boldsymbol{L}$, which are positive stable and is denoted by $\boldsymbol{M}$.

Theorem 3.1 ([7,8]). *A matrix $M \in \boldsymbol{M}$ if and only if the following equivalent statements hold:*

(s1) *M has an eigenvector $\alpha \in \omega_N$, the corresponding to it eigenvalue λ is real and such that $0 < \lambda \leq \operatorname{Re}\lambda_i(M)$, $i = 1, \dots, N$.*

(s2) *M^{-1} exists and its entries are nonnegative (nonnegative matrix).*

(s3) *There exists a vector $\beta > 0$, such that $M\beta > 0$.*

(s4) *There exists a positive diagonal matrix D, such that $M^TD + DM > 0$.*

Positive definiteness of the CM $C(x)$ in (2) is only a sufficient condition for robust stability. The next result states that under some assumption, it becomes a necessary one, as well.

Theorem 3.2. *Suppose that $MC(x) \in \boldsymbol{L}$ for all x, $M \in \boldsymbol{M}$. The following statements are equivalent:*

(i) *$v(\alpha, x)$ is a valid LF for system 1.*

(ii) *For any x there exists a vector $\beta(x) > 0$, such that $C(x)\beta(x) > 0$.*

(iii) *$C(x)$ is a positive definite CM for all x.*

Proof. Let (i) hold, i.e., $\Pi(\alpha, 2)$ in (2) is positive definite on ω_N. In accordance with the assumption that $MC(x) \in \boldsymbol{L}$ for all x, and Theorem 3.1, (s1), there exists some vector $\alpha(x) \in \omega_N$, such that $MC(x)\alpha(x) = \lambda(x)\alpha(x)$ for all x. It follows that $C(x)\alpha(x) = \lambda(x)M^{-1}\alpha(x)$ and

$$\alpha^T(x)C(x)\alpha(x) = \lambda(x)\alpha^T(x)M^{-1}\alpha(x) = -\dot{v}[\alpha(x), x] > 0$$

for all α, x, where M^{-1} is a non-negative matrix (Theorem 3.1, (s2)). The scalar $\alpha^T(x)M^{-1}\alpha(x)$ is positive, then $\lambda(x)$ is also positive, by necessity, and

$MC(x) \in \boldsymbol{M}$ for all x, or equivalently, $MC(x)\beta(x) > 0$ for all x, for some $\beta(x) > 0$ (Theorem 3.1, (s3) and (ii) follows, since $M^{-1}[MC(x)\beta(x)] = C(x)\beta(x) > 0$. If (ii) holds, i.e., $C(x)M^TM^{-T}\beta(x) = C(x)M^T\tilde{\beta}(x) > 0$, $\tilde{\beta}(x) > 0$ for all x, then $C(x)M^T \in \boldsymbol{M}$ for all x, since $C(x)M^T \in \boldsymbol{L}$ for all x. For any diagonal matrix $D > 0$, matrix $K^T(x) = D^{-1}C(x)M^TD \in \boldsymbol{M}$. In accordance with Theorem 3.1, (s4), there exist some diagonal matrices $D^2(x) > 0$ and $D > 0$, such that $D^{-1}(x)K^T(x)D(x) + D(x)K(x)D^{-1}(x) > 0$ for all x, i.e.,

$$z^T\{[D^{-1}(x)D^{-1}C(x)D^{-1}(x)][D(x)M^TDD(x)] \\ + [D(x)DMD(x)][D^{-1}(x)C(x)D^{-1}D^{-1}(x)]\}z > 0,$$

for all $z \in \boldsymbol{x}_N$ and $M^TD + DM > 0$. All eigenvalues of $D^{-1}(x)C(x)D^{-1}D^{-1}(x)$ are real and let $z = z(x)$ be the eigenvector corresponding to the minimal $\lambda(x)$. The above scalar inequality becomes

$$\lambda(x)z^T(x)[D(x)(M^TD + DM)D(x)]z(x) > 0 \text{ for all } x \Rightarrow \\ \lambda(x) = \lambda_{\min}[C(x)\tilde{D}(x)] > 0 \text{ for all } x \Leftrightarrow C(x) > 0 \text{ for all } x,$$

where $\tilde{D}(x) = [DD^2(x)]^{-1}$ is a positive diagonal matrix. Finally, (iii) always implies (i). □

Due to Theorem 3.2, the assumption $C(x) \in \boldsymbol{L}$ for all x is not necessarily required any more, in order to have statements (ii) and (iii) applicable for robust stability analysis. Although a M-matrix is used for their proof, it is not present in them. For the particular case $M = I$ and $\beta(x) = \beta_1$ one obtains the sufficient condition due to Theorem 2.1, but applied to some "external" HMP $\tilde{\Pi}(\alpha, 2) \leq \Pi(\alpha, 2)$ for all α with a corresponding to it CM $\tilde{C}(x)$, forced to be in $\boldsymbol{L}$ for all x.

Let $\alpha(s)$ denotes a vector with $s \geq 2$ arbitrarily selected entries from α. If $v(s)$ is the set of $s \times 1$ vectors with entries representing an arbitrary non-descending sequence, then all possible systems of $s!0.5s(s-1)$ pairwise inequalities $\alpha_i \leq \alpha_j$, α_i, $\alpha_j \in \alpha(s)$, $i \neq j$, are described by the set of ordered vectors $\alpha_p(s) \in v(s)$, $p = 1, \dots, s!$. For fixed N, the number of all possible distinct and compatible monomial inequalities $\alpha_i\alpha_j \leq \alpha_u\alpha_v$, $i \leq j$, $u \leq v$, $ij \neq uv$, is given by

$$\mu(s) = 0.5[s(s-1)N + \sum_{i=0}^{s-2}(s-i)(s-i-1) + \sum_{k=2}^{s-1}\sum_{i=k}^{s-1}(s-i+1)(s-i)].$$

For any $\alpha_p(s)$, the $\mu(s)$ scalar inequalities imply $\mu(s)$ matrix inequalities $(\alpha_i\alpha_j - \alpha_u\alpha_v)X_{ijuv,p} \leq 0$, where $X_{ijuv,p} = X_{uvij,p} \geq 0$ are arbitrary matrices. Consider the associated with $\alpha_p(s) \in v(s)$ HMP

$$(3) \quad \tilde{\Pi}_p(\alpha, 2) = \sum_{\substack{i,j=1 \\ i \leq j}}^{N} \alpha_i\alpha_j(\sum_{\substack{u,v=1 \\ u \leq v \\ uv \neq ij}}^{N} \mu_{ijuv,p}X_{ijuv,p}) = \sum_{\substack{i,j=1 \\ i \leq j}}^{N} \alpha_i\alpha_j\tilde{\Pi}_{ij,p}, \quad p = 1, \dots, s!,$$

where $\mu_{ijuv,p} = 1$ if $\alpha_i\alpha_j - \alpha_u\alpha_v \leq 0$, $\mu_{ijuv,p} = -1$ otherwise, and $\mu_{ijuv,p} = 0$ if the sign of the monomial difference is indefinite, due to $s < N$. Consider the HMP

(4)

$$\Pi_p(\alpha, 2) = \Pi(\alpha, 2) + \tilde{\Pi}_p(\alpha, 2) = \sum_{\substack{i,j=1 \\ i \leq j}}^{N} \alpha_i\alpha_j\Pi_{ij,p},\ \Pi_{ij,p} = \Pi_{ij} + \tilde{\Pi}_{ij,p},\ p = 1, \ldots, s!\,.$$

The next theorem provides an alternative necessary and sufficient robust stability condition.

Theorem 3.3. *$v(\alpha, x)$ is a valid LF if and only if for any $s \geq 2$ there exist $s!\mu(s)$ parameter matrices in* (3)*, such that all $s!$ HMPs in* (4) *are positive definite on ω_N.*

Proof. Let (2) hold, i.e., $\Pi(\alpha, 2) > 0$ for all α. A based on the famous Polya's theorem result, obtained in [11], states that there exists some positive integer d, such that all coefficients of the HMP $|\alpha|^d\Pi(\alpha, 2) = \Pi(\alpha, d + 2)$ of degree $d + 2$ are positive definite in this case. Then, $\Pi_p(\alpha, 2) > 0$ for all α, p iff $|\alpha|^d\Pi_p(\alpha, 2) = \Pi(\alpha, d + 2) + \tilde{\Pi}_p(\alpha, d + 2) > 0$ for all α, p. For any $s \geq 2$ and p, there always exist some appropriate $\mu(s)$ positive semi-definite matrices in (3), such that all coefficients of the HMP $|\alpha|^d\Pi_p(\alpha, 2)$ are positive definite matrices which guarantees that $\Pi_p(\alpha, 2) > 0$ for all α, p. Let the converse be true, i.e., $\Pi_p(\alpha, 2) > 0$ for all α, p, and consider an arbitrary vector α. For any $s \geq 2$ there exists some vector $\alpha_p(s)$, such that the s common entries of α and $\alpha_p(s)$ represent the same non-descending sequence. Having in mind (3) and (4), one has $\tilde{\Pi}_p(\alpha, 2) \leq 0 \Rightarrow 0 < \Pi_p(\alpha, 2) \leq \Pi(\alpha, 2)$ for this particular α, but since it has been arbitrarily chosen, it follows that $\Pi(\alpha, 2) > 0$ for all α, and inequality (2) holds. □

In other words, (2) is valid if and only if there exist $s!\mu(s)$ matrices in (3), such that

$$(5) \quad 0 < -\dot{v}(\alpha, x) + x^T[\tilde{\Pi}_p(\alpha, 2)]x = x^T\Pi_p(\alpha, 2)x = 0.5\alpha^T[C(x) + \tilde{C}_p(x)]\alpha = 0.5\alpha^T C_p(x)\alpha$$

for all α, x, p, where $\tilde{C}_p(x)$ and $C_p(x)$, $p = 1, \ldots, s!$, are the CMs for the HMPs in (3) and (4), respectively. If $s = 1$, then $v(1) \equiv \varnothing$, $\mu(1) = 0$ and (5) reduces to (2).

Consider an arbitrary matrix coefficient defined in (4) and denote

$$\Pi_{ij,p} = \Pi_{ij} + \Pi^+_{ij,p} + \Pi^-_{ij,p}, \quad \Pi^+_{ij,p} = \sum_{\mu_{ijuv,p}=1} X_{ijuv,p} \geq 0,$$

$$\Pi^-_{ij,p} = -\sum_{\mu_{ijuv,p}=-1} X_{ijuv,p} \leq 0, \quad i \leq j,\ p = 1, \ldots, s!\,.$$

Remark 3.1. For any $\alpha_p(s)$ one has $\mu_{ijuv,p} = -\mu_{uvij,p}$. This means that all sums $\Pi^+_{ij,p}$, $i \leq j$, are composed of distinct matrices (the same refers to all $\Pi^-_{ij,p}$) since, if $\mu_{ijuv,p} = 1$, then $X_{ijuv,p}$ participates only in the sum $\Pi^+_{ij,p}$ and it appears only once more time, but now as $X_{uvij,p} = -X_{ijuv,p}$, in the sum $\Pi^-_{uv,p}$. Respective conclusions are made when $\mu_{ijuv,p} = -1$.

Theorems 3.2 and 3.3 give rise to the following new robust stability conditions.

Lemma 3.1. *For an integer $s \geq 2$ and $s!\mu(s)$ parameter matrices in (3) the following statements are distinct sufficient conditions for validity of the inequality in (5):*

(i) *There exist $s!$ M-matrices M_p and positive vectors $\beta_p = (\beta_{j,p}) \in \mathbb{R}^N$, such that for all x*

$$M_p C_p(x) \in L, \quad C_p(x)\beta_p > 0, \ p = 1, \ldots, s! \quad \textit{iff}$$

$$\sum_{\substack{j=1 \\ j \geq i}}^{N} \beta_{j,p}(\pi_{ij}^{-1}\Pi_{ij,p}) > 0, \ i = 1, \ldots, N, \ p = 1, \ldots, s! \ . \tag{6}$$

(ii) *There exist $s!0.5N(N+1)$ scalars $c_{ij,p}$, such that for all x*

$$c_{ij,p}(x) \geq c_{ij,p}, \ i \leq j, \quad C_p = [c_{ij,p}] > 0, \ p = 1, \ldots, s! \ . \tag{7}$$

Proof. The proof of the first statement follows easily from Theorem 3.2, (ii), and Theorem 3.3. If (7) holds, then inequality (5) is valid, since $0 < 0.5\alpha^T C_p \alpha \leq 0.5\alpha^T C_p(x)\alpha$ for all α, x, p. □

For the particular case when $M_p = I$ for all p, and if $\Pi^-_{ij,p} \neq 0$, there always exist some parameter matrices, such that $c_{ij,p}(x) = x^T C_{ij,p} x$, $i < j$, where $C_{ij,p} \leq 0$ are arbitrary matrices, due to Remark 3.1. Therefore, there is no need to estimate these entries by some scalars $x^T V_{ij,p} x \leq 0$, for all x, as it is done in Theorem 2.1. It will be shown that for any $s \geq 2$, Lemma 3.1 provides less conservative robust stability conditions than Theorems 2.1 and 2.2. Let V_{ij} are arbitrary matrices satisfying the assumptions of Theorem 2.1 and $\Pi^-_{ij,p} = V_{ij} - \Pi_{ij} \leq 0$ for all $i \leq j$ and p if $\Pi^-_{ij,p} \neq 0$. This particular choice guarantees that $\Pi_{ij,p} = V_{ij} + \Pi^+_{ij,p} \geq V_{ij}$ for all $i \leq j$ and p. Then, for $M_p = I$, $\beta_p = \beta_1$ for all p, the matrix inequalities in (6) hold, by necessity. Suppose now that the defined in Theorem 2.2 matrix $C = [c_{ij}]$ is positive definite. Applying the same arguments, it can be easily shown that for any matrix C, there always exist some appropriate parameter matrices in (3), such that the entries of the CMs $C_p(x)$ in (5) and C_p in (7), satisfy the inequalities $c_{ij,p}(x) \geq c_{ij,p} \geq c_{ij}$ for all $i \leq j$ and p. For $c_{ij,p} = c_{ij}$ for all $i < j$ and p, one has $C_p \geq C$ for all p. If the robust stability of system (1) is ensured via Theorem 2.1 (2.2), then at least one of the statements in Lemma 3.1 holds as well, but the converse is not true in general. As the dimension of the selected vector $\alpha(s)$ increases, one obtains more and more relaxed conditions and the maximal effect is achieved when the whole vector α is selected, i.e., for $s = N$. The next lemma states a condition, which eliminates the dependence of all CMs in (5) on vector x, without estimations.

Lemma 3.2. *Let α_m denote the minimal entry of the p-th ordered vector $\alpha_p(N)$, $p = 1, \ldots, N!$. Consider the $N!\mu(N)$ parameter matrices in (3), chosen for any p, as follows:*

$$(8)\quad -\Pi^-_{ij,p} = \Pi_{ij} + \Pi^+_{ij,p} \text{ if } \Pi_{ij} + \Pi^+_{ij,p} \geq 0 \Rightarrow c_{ij,p}(x) = c_{ij,p} = 0,\ i < j,$$

$$(9)\quad -\Pi^-_{ij,p} = \Pi_{ij} + \Pi^+_{ij,p} - \lambda_{\min}(\Pi_{ij} + \Pi^+_{ij})I \geq 0 \text{ if } \lambda_{\min}(\Pi_{ij} + \Pi^+_{ij}) = c_{ij,p} < 0$$
$$\Rightarrow c_{ij,p}(x) = c_{ij,p} < 0,\ i < j,$$

$$(10)\quad -\Pi^-_{ii,p} = \Pi_{ii} + \Pi^+_{ii,p} - \lambda_{\min}(\Pi_{ii} + \Pi^+_{ii})I \geq 0 \Rightarrow$$
$$c_{ii,p}(x) = 2\lambda_{\min}(\Pi_{ii} + \Pi^+_{ii,p}) = c_{ii,p},\ i \neq m.$$

For this choice, (5) holds iff $C_p = [c_{ij,p}] > 0$ for all p, where $0.5N(N+1)-1$ of the entries of C_p are defined in (8), (9) and (10) and $c_{mm,p} = 2\lambda_{\min}(\Pi_{mm} + \Pi^+_{mm,p})$ for all p.

Proof. Having in mind Remark 3.1, all equalities in (8), (9) and (10) are possible, since $s = N$, and $\mu_{ijuv,p} = -\mu_{uvij,p} \neq 0$ for all p. For any p the entries of the CM $C_p(x)$ are equal to some scalars $c_{ij,p}$ except for the entry $c_{mm,p}(x) = 2x^T(\Pi_{mm}+\Pi^+_{mm,p})x$ since $\mu_{mmuv,p} = 1$ for all u, v, i.e., $\Pi^-_{mm,p} = 0$ for all p. Taking $c_{mm,p}$ as above is obligatory, since $c_{mm,p}(x) = c_{mm,p}$ for all p, is admissible. Then, inequality (5) holds if and only if $\alpha^T C_p \alpha > 0$ for all α, p. From Theorem 3.2, (iii), it follows that this condition is equivalent to $C_p > 0$ for all p, since $C_p \in L$ for all p. □

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Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria