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MATHEMATIQUES

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MODELS FOR FUZZY MULTICRITERIA DECISION MAKING BASED ON FUZZY RELATIONS

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Abstract

The paper presents models and corresponding algorithms for solving fuzzy multicriteria decision making problems. The models use or transform the initial information to fuzzy preference relations by each criterion. These relations possess required properties to solve the problems of choice or ordering of the alternatives. The weights of the criteria are real numbers or weighting functions.

Key words: multicriteria decision making, fuzzy relations, aggregation operators, fuzzy relations' properties, weighting function

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1. Introduction. The multicriteria decision making models in fuzzy environment are based on:

- a finite set of alternatives, among which a decision maker has to choose (choice problem), or to rank (ranking problem), or to part (cluster problem);
- a finite set of judges or criteria on the base of which the alternatives are evaluated;
- a criteria importance, i.e. weights of the criteria.

The alternatives in decision making problems are usually evaluated from different points of views that correspond to particular criteria. The criteria can be quantitative and qualitative ones. Usually quantitative criteria are assessed by means of crisp numerical values. The qualitative criteria are presented in

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qualitative terms by means of linguistic variables. The weights of the criteria can be crisp or fuzzy.

The solution scheme of the multicriteria decision making problems basically consists of three phases:

A. A uniform phase. It is required to make the information uniform if the criteria are in different scales. One basic approach to make this is to use fuzzy relations over the set of alternatives as the main element of uniform representation. Therefore, some transformation functions are needed to define the relations between the couple of alternatives by each criterion. These transformation functions define relations with different properties, for example, similar or preference relations. It is more realistic to use fuzzy relations because they may model situations, whenever interactions between the alternatives are not exactly determined, e.g. they can hold uncertainties of the kind "similar", "close", "more effective", "more preference" and so on. Besides that they reflect the interests of the experts or the decision-maker. The fuzzy relations and their properties are investigated by many authors.

B. An aggregation phase of the performance values with respect to all criteria for obtaining a union performance value for the alternatives. A purposeful approach for uniting individual evaluations corresponding to an alternative is to use aggregation operators. There is a large range of operators, which can be advantageously used in the confluence of the criteria. The choice of an operator for specific application depends on various factors. Some choice has to be made according to, e.g.:

- the mathematical model of the operators;
- the properties of the operators for deciding problems of ranking or choice, or clustering of the alternatives' set;
- the sensitivity of the operators for small variations of their arguments.

A very good overview of the aggregation operators, by presenting the characteristics, the advantages and disadvantages of each operator and the relations between them, is available in [2]. The dependence between the properties of the aggregated relation and the properties of the individual relations by each criterion for some operators is investigated in [3,7]. The sensitivity of the operators with respect to variations in their arguments is defined and computed in [7].

The weighted aggregation is very important in decision problems. The weights present or not present in the models of the operators. In the second case, weighted transformations in these operators are performed to use the criteria importance in decision making problems [8].

C. An exploitation phase of the union performance value for obtaining a rank ordering, sorting or choosing the alternatives. The problems of choice of a subset from the "best" in some sense alternatives; ordering over the whole set of alternatives; partition the set of alternatives of the subsets from the similar, close ones, i.e. partition from clusters, have to be solved in this phase.

The models for multicriteria decision making depend on the kind of the criteria, the kind of their weights (importance) and the methods for achieving the aims of the problems. Let $A = \{a_1, \dots, a_i, \dots, a_n\}$ be a set of alternatives, evaluated by the criteria $K = \{k_1, \dots, k_j \dots, k_m\}$. The criteria weights (importance) are $W = \{w_1, \dots, w_j, \dots, w_m\}$. The model in a matrix form is

Alternatives/Criteria	k_1	$\cdots$	k_j	$\cdots$	k_m
a_1	x_{11}	$\cdots$	x_{1j}	$\cdots$	x_{1m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_i	x_{i1}	$\cdots$	x_{ij}	$\cdots$	x_{im}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_n	x_{n1}	$\cdots$	x_{nj}	$\cdots$	x_{nm}

where x_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$ is the estimation of the alternative a_i by the criterion k_j.

2. Models by criteria with weights – real numbers. *2.1. Uniform phase.* The information about the alternatives can be supplied in different scales. In this case, the information uniform is made using fuzzy relations. Therefore, it needs some transformation functions that define the relations between the couple of alternatives by each criterion. When the alternatives' number n is not very large then a fuzzy relation by the criterion k_k can be present as an $n \times n$ matrix, i.e. $R_k = \left\| r_{ij}^k \right\|$, where $r_{ij}^k = \mu_k(a_i, a_j)$, $i, j = 1, \dots, n$, $k = 1, \dots, m$, and $\mu_k : A \times A \to [0, 1]$ is the membership function of the relation R_k and r_{ij}^k is the membership degree to R_k given from the expert or computed from the transformation function by comparing of the couple of alternatives a_i and a_j by criterion k_k. One has

$$R_k = \begin{pmatrix} \mu_k(a_1,a_1) & \dots & \mu_k(a_1,a_j) & \dots & \mu_k(a_1,a_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_k(a_i,a_1) & \dots & \mu_k(a_i,a_j) & \dots & \mu_k(a_i,a_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_k(a_n,a_1) & \dots & \mu_k(a_n,a_j) & \dots & \mu_k(a_n,a_n) \end{pmatrix} = \begin{pmatrix} r_{11}^k & \dots & r_{1j}^k & \dots & r_{1n}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{i1}^k & \dots & r_{ij}^k & \dots & r_{in}^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n1}^k & \dots & r_{nj}^k & \dots & r_{nn}^k \end{pmatrix},$$

where $k = 1, \dots, m$.

The transformation function suggested in [6] is used in considered here models to make the information uniform. The preference degree of the couple of alternative a_i before a_j for criterion k_k is defined as

$$\mu_k(a_i, a_j) = \begin{cases} 1 & \text{if } i = j, \\ 0.5 + \dfrac{x_{ik} - x_{jk}}{2(\max\limits_i\{x_{ik}\} - \min\limits_i\{x_{ik}\})} & \text{if } i \neq j, \end{cases}$$

where x_{ik}, x_{jk}, $i, j = 1, \dots, n$, $k = 1, \dots, m$ are some real values. It is proved that the obtained fuzzy relation is reflexive ($r_{ii}^k = 1$), additive reciprocal ($r_{ij}^k + r_{ji}^k = 1$) and max-min transitive ($r_{ij}^k \geq \min(r_{is}^k, r_{sj}^k)$, $s = 1, \dots, n$), i.e. it is fuzzy total ordering [5].

2.2. Aggregation phase. The obtained fuzzy relations by all criteria are fused using the following aggregation operators.

2.2.1. Aggregation operators with weighted coefficients.

- Weighted Mean (WMean) [13]:

$$\mu(a_i, a_j) = \sum_{k=1}^{m} w_k \mu_k(a_i, a_j) \text{ when } 0 \leq w_k \leq 1, \quad \sum_{k=1}^{m} w_k = 1.$$

- Weighted Geometric (WGeom) [1]:

$$\mu(a_i, a_j) = \prod_{k=1}^{m} [\mu_k(a_i, a_j)]^{w_k} \text{ when } 0 < w_k \leq 1, \quad \sum_{k=1}^{m} w_k = 1.$$

- Weighted MaxMin (WMaxMin) [4], the aggregated membership function of which expresses the relation "a_i is not worse than a_j for at least one weighted criterion":

$$\mu(a_i, a_j) = \max_k\{\min(\mu_k(a_i, a_j), w_k)\}$$

  where $0 \leq w_k \leq 1$, $\max\limits_k\{w_k\} = 1$, $k = 1, \dots, m$.

- Weighted MinMax (WMinMax) [4], the aggregated relation of which means that "a_i is not worse than a_j for all weighted criteria":

$$\mu(a_i, a_j) = \min_k\{\max(\mu_k(a_i, a_j), w_k)\}$$

  where $0 \leq w_k \leq 1$, $\max\limits_k\{w_k\} = 1$, $k = 1, \dots, m$.

2.2.2. Aggregation operators without weighted coefficients. The following well-known operators are used:

- MaxMin operator [13]: $\mu(a_i,a_j)=\alpha\max_k\{\mu_k(a_i,a_j)\}+(1-\alpha)\min_k\{\mu_k(a_i,a_j)\}$, $\alpha\in[0,1]$, $i,j=1,\dots,n$, $k=1,\dots,m$.
- MinAvg operator [13]: $\mu(a_i,a_j)=\frac{\lambda}{m}\sum_{k=1}^{m}\mu_k(a_i,a_j)+(1-\lambda)\min_k\{\mu_k(a_i,a_j)\}$, $\lambda\in[0,1]$, $i,j=1,\dots,n$.
- Gamma operator [13], defined for $\gamma\in[0,1]$

$$\mu(a_i,a_j)=\begin{cases}\left[\prod_{k=1}^{m}\mu_k(a_i,a_j)\right]^{1-\gamma}\left[1-\prod_{k=1}^{m}(1-\mu_k(a_i,a_j))\right]^{\gamma} & \text{if } \mu_k(a_i,a_j)\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

As one can see, the weight coefficients are not present in these mathematical models of these operators. How can one use the weights in the cases? Taking into account the conception in [14] the membership degrees in these operators may be transformed using weights as follows:

$$g(w_k,\mu_k(a_i,a_j))=\tilde{\mu}_k(a_i,a_j),\quad a_i,a_j\in A,\ k=1,\dots,m.$$

Then, the weighted aggregation is obtained as

$$\mathrm{Agg}(\tilde{\mu}_1(a_i,a_j),\dots,\tilde{\mu}_m(a_i,a_j))=\mu^w(a_i,a_j),$$

where Agg denotes an aggregation operator and the function g satisfies defined properties [14]. The following weighted transformations for operators Min and Max are used in these models, respectively

$$g(h(w_k),\mu_k(a_i,a_j))=S(1-w_k,\mu_k(a_i,a_j)),$$

or

$$g(h(w_k),\mu_k(a_i,a_j))=T(w_k,\mu_k(a_i,a_j)),$$

where T, S are t-norm and corresponding t-conorm, h is a linear function. Therefore,

$$\min(\mu_1^w(a_i,a_j),\dots,\mu_m^w(a_i,a_j))=\min\{S(1-w_1,\mu_1(a_i,a_j)),\dots,S(1-w_m,\mu_m(a_i,a_j))\},$$
$$\max(\mu_1^w(a_i,a_j),\dots,\mu_m^w(a_i,a_j))=\max\{T(w_1,\mu_1(a_i,a_j)),\dots,T(w_m,\mu_m(a_i,a_j))\}.$$

3. Models for criteria with importance – weighting functions. The introduction of weighting functions depending continuously on criterion satisfaction values produces weighted aggregation operators with complex dependency from these values. These functions have to be monotonic and sensitive [12], as well. Let $f_1(x), \ldots, f_m(x)$, $x \in [0,1]$ be some real functions with defined properties given as importance of the criteria. The continuous functions $f_k(x)$, $k = 1, \ldots, m$ are defined in the unit interval for each $x \in [0,1]$ and they have continuous derivatives $f'_k(x)$, $k = 1, \ldots, m$ in this interval. The following fitting weighting functions [12] are considered in the proposed model:

- Linear weighting functions $f_k(x) = 1 + \beta_k x$ with parameters $0 \le \beta_k \le 1$, $k = 1, \ldots, m$, $m \ge 2$;

- Parametric linear weighting functions

$$f_k(x) = \alpha_k \frac{1 + \beta_k x}{1 + \beta_k} = \gamma_k (1 + \beta_k x),$$
$$0 < \alpha_k \le 1, \quad 0 \le \beta_k \le 1, \quad \gamma_k = \frac{\alpha_k}{1 + \beta_k}, \quad k = 1, \ldots, m;$$

- Quadratic weighting functions $f_k(x) = 1 + (\beta_k - \gamma_k)x + \gamma_k x^2$, $\beta_k \ge 0$, $\gamma_k \ge 0$, $k = 1, \ldots, m$.

Let the membership degrees comparing the alternatives $a_i, a_j \in A$, $i, j = 1, \ldots, n$ to fuzzy relations $R_1, \ldots, R_k, \ldots, R_m$ are $\mu_1(a_i, a_j) = x^1_{ij}, \ldots, \mu_k(a_i, a_j) = x^k_{ij}, \ldots, \mu_m(a_i, a_j) = x^m_{ij}$. The Generalized mixture operator with the following mathematical model is defined in [12]:

$$\mu^w(a_i, a_j) = \text{Agg}(\mu^w_1(a_i, a_j), \ldots, \mu^w_m(a_i, a_j)) = \begin{cases} 1 & a_i = a_j, \\ \dfrac{\sum_{k=1}^m f_k(x^k_{ij}) x^k_{ij}}{S(a_i, a_j)} & a_i \ne a_j, \end{cases}$$

where $S(a_i, a_j) = \sum_{k=1}^m f_k(x^k_{ij})$, and $f_k(\cdot)$ is one of the weighting functions given above. It is proved that the sufficient condition for strict monotonicity of the Generalized mixture operator, i.e. to be aggregation operator, is

$$0 \le f'_k(x) \le f_k(x), \quad k = 1, \ldots, m, \quad x \in [0,1].$$

The properties of the aggregated relations obtained by the Generalized mixture operator taking into account the above weighting functions are investigated and proved in [9–11].

4. Exploitation phase – decision making. The aggregated preference relations R obtained at the end of the aggregation process do not always present any ordering properties (except reflexivity), having in mind mostly the transitivity. According to the connections with the properties of R_j, $j = 1, \ldots, m$, the aggregated relations R get by the operators given above are fuzzy preorders [7,8], i.e. they are reflexive ($r_{ii} = 1$, $i = 1, \ldots, n$) and max-Δ transitive ($r_{ij} \geq \max(0, r_{is} + r_{sj} - 1)$, $s = 1, \ldots, n$). The antisymmetrized fuzzy preorder relations R' [5] of the aggregated relations R are computed then, i.e. if $\mu(a_i, a_j) \geq \mu(a_j, a_i)$, then $\mu_{R'}(a_i, a_j) = \mu(a_i, a_j)$ and $\mu_{R'}(a_j, a_i) = 0$. R' may be represented as a triangular matrix. The graph corresponding to this matrix has no cycle and shows the fuzzy ordering of the alternatives.

5. Algorithms realized from the models. The suggested above models are developed to the algorithms and programmes that work in a real circumstances. The first algorithm transforms the initial information to the fuzzy relations by the criteria, aggregates these relations to the aggregated one using the operators with weighting coefficients, computes the antisymmetrized fuzzy preorder relation for chosen operator and presents the obtained fuzzy ordering. The second algorithm uses the aggregation operators without weighting coefficients, but if the weights have to be used, the weighting transformations are made by the help of t-norms and t-conorms. Then, it follows the steps of the first algorithm. The third algorithm uses the weighting functions considered above, computes the aggregated relations by the help of the Generalized mixture operator and gives the fuzzy ordering on the base of the obtained antisymmetrized fuzzy preorder relation.

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