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MATHEMATIQUES

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FUZZY CRITERIA IMPORTANCE DEPENDING ON MEMBERSHIP DEGREES OF FUZZY RELATIONS

Vania Peneva, Ivan Popchev

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Abstract

The problem is connected with the weighted aggregation of fuzzy relations given by criteria that evaluate a finite set of alternatives. The weights of these criteria are quadratic weighting functions depending on the membership degrees of the fuzzy relations. Transformed membership degrees computed by the help of these weighting functions are used in aggregation procedures. The properties of these relations required to decide the problems of choice, ranking or clustering of the alternatives' set are proved.

Key words: fuzzy multicriteria decision making, fuzzy relations, aggregation operators, fuzzy relations' properties, weighting functions

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1. Introduction. The following multicriteria decision making problem is considered. A finite number of alternatives is evaluated by several independent fuzzy criteria comparing each couple of the alternatives by assigning fuzzy relations with membership degrees values in the unit interval. The importance of each criterion is quadratic weighting function with values in unit interval as well. Aggregation operators, uniting the membership degrees by all relations, taking into account the respective weighting functions in order to compute the multicriteria score of the fuzzy relations are used. The purpose is the obtained aggregated fuzzy relation to possess some properties giving a possibility to decide some multicriteria decision making problems.

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The usage of weighting functions that depend continuously on the criterion satisfaction values (i.e. good or bad criteria performances) instead of constant weights is suggested in [8–10]. Models for computation of weighted fuzzy relations with different linear weighting functions and their advisability are considered in [6, 7]. Quadratic weighting functions will be used in this investigation as some other variant of such functions.

Let $A = \{a, b, c, \ldots, n\}$ be a finite number of alternatives evaluated by fuzzy criteria $K = \{k_1, \ldots, k_m\}$. Let $R_1, \ldots, R_i, \ldots, R_m$ be the matrices of the fuzzy relations corresponding to the different criteria, i.e. $R_i = \|\mu_i(a, b)\|$, $i = 1, \ldots, m$, $m \geq 2$, $\forall a, b \in A$, where $\mu_i(a, b) \in [0, 1]$ is the membership degree of the comparison of a, $b \in A$ for the fuzzy relation R_i (R_i means a fuzzy relation and a matrix corresponding to this relation as well, for simplicity). The weights of the criteria are given as functions $w_1(R_1), \ldots, w_m(R_m)$ of the membership degrees to each relation.

According to [11], each of the membership degrees may be transformed using weighting functions of the criteria as follows:

$$(1) \qquad \mu_i^w(a, b) = t(w_i(\mu_i(a, b)), \mu_i(a, b)), \; a, b \in A, \; i = 1, \ldots, m,$$

and then the weighted aggregation is obtained as

$$(2) \qquad \mu^w(a, b) = \mathrm{Agg}(\mu_1^w(a, b), \ldots, \mu_i^w(a, b), \ldots, \mu_m^w(a, b)),$$

where Agg denotes some aggregation operator [2] and the function $t(w, x)$ satisfies the following properties [11]:

$$x > y \rightarrow t(w, x) \geq t(w, y); \quad t(w, x) \text{ is monotone in } w; \; t(0, x) = id, \; t(1, x) = x,$$

with the identity element, id, such that it does not change the aggregated value by adding it to the aggregation. The form of t depends on the type of aggregation performed, e.g. it may be t-norm or t-conorm [4,5,11], taking into account their properties. Here will be discussed the properties of the new relations R_i^w, $i = 1, \ldots, m$ with membership functions $\mu_i^w(a, b)$, $a, b \in A$, $i = 1, \ldots, m$, where $w_i(R_i)$, $i = 1, \ldots, m$ are quadratic weighting functions, t is the product t-norm and Agg is the Weighted Mean operator.

2. Quadratic weighting functions. Let $\mu_1(a, b) = x_1, \mu_2(a, b) = x_2, \ldots, \mu_m(a, b) = x_m$ be the membership degrees obtained from the comparison of the alternatives $a, b \in A$ to the fuzzy relations $R_1, \ldots, R_i, \ldots, R_m$. The membership function of the Weighted Mean operator is defined as [2]

$$(3) \qquad \mu^w(a, b) = \sum_{i=1}^{m} w_i(x_i) x_i = \sum_{i=1}^{m} \mu_i^w(a, b),$$

with weighting functions in this case

$$(4) \quad w_i(x_i) = \frac{g_i(x_i)}{\sum_{i=1}^{m} g_i(x_i)} = \frac{g_i(x_i)}{S(a, b)} \text{ and the normalization condition } \sum_{i=1}^{m} w_i(x_i) = 1.$$

Consider the following functions:

$$g_i(x) = 1 + (\beta_i - \gamma_i)x + \gamma_i x^2,\ \beta_i \geq 0,\ \gamma_i \geq 0\ [8],\ \text{i.e.} \tag{5}$$

$$\mu^w(a,b) = \frac{\sum_{i=1}^{m} g_i(x_i)x_i}{\sum_{i=1}^{m} g_i(x_i)} = \frac{\sum_{i=1}^{m} g_i(x_i)x_i}{S(a,b)}. \tag{6}$$

The linear weighting functions investigated in [6,8] correspond to $\gamma_i = 0$.

It is assumed that the convex quadratic functions $g_i(x)$, $i = 1, \dots, m$ are defined in the unit interval; they are continuous and have continuous derivatives $g_i'(x)$, $i = 1, \dots, m$ in this interval. It is proved [8] that the sufficient condition for the strict monotonicity of the operator (6) is $g_i'(x) \leq g_i(x)$, $i = 1, \dots, m$, $x \in [0,1]$. The above conditions are performed if [8]

$$0 \leq \gamma_i \leq 1,\ \gamma_i \leq \beta_i \leq \beta_c(\gamma_i),\ i = 1, \dots, m,\ \text{with} \tag{7}$$

$$\beta_c(\gamma_i) = \begin{cases} \gamma_i + 1 & \text{for } 0 \leq \gamma_i \leq 0.5 \\ \gamma_i + 2\sqrt{\gamma_i(1-\gamma_i)} & \text{for } 0.5 \leq \gamma_i \leq 1 \end{cases}.$$

3. Properties of the transformed weighted relations (1) with weighting functions (5). Introduce the following notations for simplicity:

$$\mu_i(a,c) = z_i,\ \mu_i(a,b) = x_i,\ \mu_i(b,c) = y_i, \tag{8}$$

$$\begin{aligned} S(a,b) &= \sum_{i=1}^{m} g_i(\mu_i(a,b)) = \sum_{i=1}^{m} (1 + (\beta_i - \gamma_i)x_i + \gamma_i x_i^2) = m + \sum_{i=1}^{m} X_i \\ S(b,c) &= \sum_{i=1}^{m} g_i(\mu_i(b,c)) = \sum_{i=1}^{m} (1 + (\beta_i - \gamma_i)y_i + \gamma_i y_i^2) = m + \sum_{i=1}^{m} Y_i \\ S(a,c) &= \sum_{i=1}^{m} g_i(\mu_i(a,c)) = \sum_{i=1}^{m} (1 + (\beta_i - \gamma_i)z_i + \gamma_i z_i^2) = m + \sum_{i=1}^{m} Z_i. \end{aligned} \tag{9}$$

The reflexivity, symmetry and transitivity properties of the relations R_i^w, $i = 1, \dots, m$ with membership functions

$$\mu_i^w(a,b) = \begin{cases} 1 & \text{if } a = b \\ w_i(x_i)x_i = \dfrac{g_i(x_i)x_i}{S(a,b)} & \text{if } a \neq b \end{cases} \quad \forall a,b \in A,\ i = 1, \dots, m,\ m \geq 2, \tag{10}$$

where $g_i(x)$, $i = 1, \dots, m$ are the functions (5), will be studied. It is easy to see that these weighted relations preserve the reflexive and symmetrical properties of the initial relations. But the property of max-min transitivity does not preserve as it may be verify from simple examples.

Proposition 3.1. If the fuzzy relations $R_i = \|\mu_i(a,b)\|$, $i = 1,\dots,m$, $\forall a,b \in A$ are max-min transitive, then the relations R_i^w, $i = 1,\dots,m$ with membership functions (10) are max-Δ transitive fuzzy relations.

Proof. Let R_i be max-min transitive relations, i.e.,

$$\mu_i(a,c) \geq \min(\mu_i(a,b), \mu_i(b,c)), \quad \forall a,b,c \in A, \ i = 1,\dots,m.$$

According to (8) the above inequality becomes

$$z_i \geq \min(x_i, y_i), \ i = 1,\dots,m. \tag{11}$$

It has to be proved that

$$\mu_i^w(a,c) \geq \max(0, \mu_i^w(a,b) + \mu_i^w(b,c) - 1), \ \forall a,b,c \in A, \ i = 1,\dots,m, \tag{12}$$

or more precisely

$$\frac{g_i(z_i)z_i}{S(a,c)} \geq \max(0, \frac{g_i(x_i)x_i}{S(a,b)} + \frac{g_i(y_i)y_i}{S(b,c)} - 1). \tag{13}$$

Let $j \in [1,m]$ and $x_j \leq z_j \leq y_j$, e.g. from (11). If it can be proved that

$$\frac{g_j(z_j)z_j}{S(a,c)} \geq \frac{g_j(x_j)x_j}{S(a,b)} \frac{g_j(y_j)y_j}{S(b,c)}, \tag{14}$$

then (13) is valid in view of the inequality $z \geq xy \geq \max(0, x+y-1)$, $x,y,z \in [0,1]$.

From $x_j \leq z_j \leq y_j$ it follows that $g_j(z_j)z_j \geq g_j(x_j)x_j$.

If $S(a,c) \leq S(a,b)$, $S(a,c) \leq S(b,c)$ or $S(a,c) \leq S(a,b)S(b,c)$, then (14) is valid.

Let $S(a,c) > S(a,b)$, $S(a,c) > S(b,c)$ and $S(a,c) > S(a,b)S(b,c)$, then according to (14) it has to be proved that

$$\frac{1}{S(a,c)} \geq \frac{g_j(y_j)y_j}{S(a,b)S(b,c)} \Rightarrow S(a,b)S(b,c) \geq S(a,c)g_j(y_j)y_j. \tag{15}$$

This inequality may be rewritten taking into account (9) as follows:

$$\left(m + \sum_{i=1}^{m} X_i\right)\left(m + \sum_{i=1}^{m} Y_i\right) \geq g_j(y_j)y_j\left(m + \sum_{i=1}^{m} Z_i\right).$$

But $m^2 \geq m g_j(y_j)y_j$, because $m \geq 2$, while $(1 + (\beta_j - \gamma_j)y_j + \gamma_j y_j^2)y_j \leq 1 + \beta_j \leq 2$ and

$$m\left(\sum_{i=1}^{m} X_i + \sum_{i=1}^{m} Y_i\right) \geq g_j(y_j)y_j \sum_{i=1}^{m} Z_i,$$

that follows from $x_i \leq z_i \leq y_i$ or $y_i \leq z_i \leq x_i$ by assumption. Therefore (15) is valid.

The properties reflexivity, symmetry and max-Δ transitivity of the relations R_i^w, $i = 1, \ldots, m$ with membership functions (10) are useful, because they give a possibility to decide the problems of choice, ranking or clustering after the computation of the aggregated relation with membership function (6). The following propositions have to be taken into account for this purpose.

Proposition 3.2. If R_i, $i = 1, \ldots, m$ are reflexive and max-min transitive relations, then the relations R_i^w, $i = 1, \ldots, m$ with membership functions (10) are fuzzy preorders (reflexive and max $-\Delta$ transitive relations).

The proof follows from Proposition 3.1 and a definition of a fuzzy preorder given in [3].

Proposition 3.3. If R_i, $i = 1, \ldots, m$ are similarity (reflexive, symmetrical and max-min transitive) relations, then the relations R_i^w, $i = 1, \ldots, m$ with membership functions (10) are likeness (reflexive, symmetrical and max-Δ transitive) relations.

The proof follows from Proposition 3.1 and a definition given in [3].

Proposition 3.4. [4] If the relations R_i^w, $i = 1, \ldots, m$ are fuzzy preorders, then the aggregated relation with the membership function (6) is a fuzzy preorder relation as well.

Proposition 3.5. [4] If R_i^w, $i = 1, \ldots, m$ are likeness relations, then the aggregated relation with the membership function (6) is a likeness relation as well.

4. Conclusion. The considered quadratic weighting functions instead of constant weights of the fuzzy criteria produces weighted aggregation with complex dependency on the membership degrees of the initial relations. The advantage of these functions is in their ability to fine the small values and to reward the great values of the membership degrees. The proved properties of the weighted relations give a possibility to use transformed relations in aggregation procedures for solving some multicriteria decision making problems. The aggregated preorder relation may be used in the problems for choice or ordering among the set of alternatives. The property of likeness of the aggregated relation is useful for solving a clustering problem of the alternatives' set.

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Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria