SOLUTION BOUNDS VALIDITY FOR CALE EXTENSION VIA SINGULAR VALUE DECOMPOSITION APPROACH

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Abstract

Motivated by the fact that upper solution bounds of the continuous Lyapunov equation are valid under some very restrictive condition, an attempt was made to extend the set of Hurwitz matrices of which such bounds are applicable. It is shown that the matrix set for which solution bounds are available is only a subset of another stable matrices set. This helps to loosen the validity restriction.

Key words: continuous Lyapunov equation, Hurwitz matrix, singular value decomposition, solution upper bounds

Introduction. The continuous algebraic Lyapunov equation (CALE) has been widely used in engineering theory. Although there are many numerical algorithms to obtain a solution, sometimes only its estimate is needed. For instance, one can estimate a stability margin for real polynomials using some of the available bounds [6]. An estimate-based approach to study robust stability and performance analysis of uncertain stochastic systems was suggested in [1]. Lyapunov equations have been used in atmospheric science applications [9], where a distinguishing property of the coefficient matrix A is its large dimension which leads to impossibility for direct solution. Therefore, for practical purposes it is desirable to determine reasonable estimates.

The estimation problem for the CALE has attracted considerable attention in the past three decades. Excellent summaries on this topic were given in [4, 5] comprising various lower and upper bounds for the largest eigenvalue, the trace, the determinant, the solution itself, etc. In practical applications, especially for stability analysis, upper bounds are desired.

In most cases, the existing upper bounds are valid under some, unfortunately very restrictive assumptions, e.g. $A^T + A$ must be negative definite. Since stability of A does not guarantee this requirement, the respective estimates are useless for a large set of stable matrices.

This fact motivated the authors of this paper to investigate the conditions under which it is possible to have valid upper bounds for solution of the CALE in cases when the existing bounds do not hold.

In other words, the results presented here extend the set of stable matrices A for which upper bounds for the largest eigenvalue, the trace and the solution matrix are valid.

This paper is organized as follows. The best known solution upper scalar bounds, when $A^T + A$ is a negative definite matrix and a recent result (based on the usage of similarity transformation) guaranteeing upper bounds validity for any Hurwitz matrix A are recalled in Section 2. Section 3 contains the main results consisting in extending the set of Hurwitz matrices for which upper matrix, maximum eigenvalue and trace bounds for the solution are valid (Theorem 3.1 and Lemma 3.2). This is achieved via the singular value decomposition of matrix A.

2. Notations and preliminaries. In what follows, the given below notations will be used. H is the set of Hurwitz (negative stable) matrices, $A_s = A^T + A$ and $A > B$, $A \geq B$ means that $A - B$ is a positive definite and positive semidefinite matrix, respectively. The eigenvalues (when real) of an $n \times n$ matrix A are denoted by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(A_s) \equiv \mu(A)$. The maximum and minimum singular values of a matrix A are $\sigma_1(A)$ and $\sigma_n(A)$, respectively. The maximum real part of the eigenvalues of a matrix A is $\kappa(A)$ and I is the identity matrix.

Consider the CALE

$$(1) \qquad\qquad A^T P + PA + Q = 0, \ A \in \mathsf{H}, \ Q > 0$$

with respect to the unknown (positive definite) matrix P. If $\mu(A) < 0$, the best known upper scalar solution bounds are:

$$(2) \qquad\qquad \lambda_1(P) \leq l_0 = \lambda_1(-QA_s^{-1}) \quad [10]$$

$$(3) \qquad\qquad \mathrm{tr}(P) \leq t_0 = -\sum_1^n [\lambda_i(Q)/\lambda_i(A_s)] \quad [3].$$

Let T be a nonsingular matrix, denote $A(T) = TAT^{-1}$ and define the matrix set

$$\mathsf{H}^- \equiv \{A(T) : \mu[A(T)] < 0\}.$$

Equation (1) can be written in a modified form as

$$(4) \qquad A^T(T)\widetilde{P} + \widetilde{P}A(T) + \widetilde{Q} = 0, \ \widetilde{P} = T^{-T}PT^{-1}, \ \widetilde{Q} = T^{-T}QT^{-1}.$$

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For any given matrix $A \in \mathsf{H}$, $A \notin \mathsf{H}^-$, there exists matrix T such that $A(T) \in \mathsf{H}^-$. This well-known fact is used in [1] to apply bounds (2), (3) for $\widetilde{P}$ in (4) and then to get estimates for P in (1).

It is also possible to get a matrix bound. Since $A(T) \in \mathsf{H}^-$, using Lyapunov's stability argument, it can be easily shown that

$$(5) \qquad P \leq \eta T^T T, \; \eta = \lambda\{-Q[(T^T T A)_s]^{-1}\}$$

and then scalar bounds for $\lambda_1(P)$ and $\mathrm{tr}(P)$ are easily obtained.

The estimation problem for P has three important aspects: (i) restrictions on matrix A, (ii) computational burden and (iii) tightness of the bounds. Bounds based on equation (4) eliminate problem (i), but require the determination of matrix T, the selection of which to obtain the tightest bound is an open and difficult question.

Matrix T is obtained by some additional computational procedure and in this sense $\bar{P} = T^T T$, $\mu(\bar{P}A) < 0$, is said to be an external Lyapunov matrix (ELM) for A. An internal Lyapunov matrix (ILM) is a matrix which can be defined entirely in terms of A. E.g., if $A \in \mathsf{H}^-$, then $\bar{P} = A^T A$ is an ILM for A.

This paper is an attempt to overcome to a certain extent the above-mentioned difficulties concerning bounds based on ELM. This is closely related with the definition of an extension $\widetilde{\mathsf{H}}$ of the conservative set H^- in sense that, if $A \notin \mathsf{H}^-$, but $A \in \widetilde{\mathsf{H}}$, there exists ILM for A. This will help to avoid the computation of an ELM which, as the order of A increases, may cause difficulties comparable with the solution of the CALE itself and thus make the respective bounds practically inapplicable.

3. Main results. Using the singular value decomposition (s.v.d.) of the coefficient matrix in (1), i.e. $A = U\Sigma V^T$, $UU^T = VV^T = I$ and Σ is a positive diagonal matrix, it is always possible to present it as a product of two matrices, i.e.

$$A = FP_1 = P_2 F, \quad F = UV^T, \quad P_1^2 = A^T A, \quad P_2^2 = AA^T.$$

Denote

$$(6) \qquad A(T) = F(T)P_1(T) = P_2(T)F(T).$$

Define the matrix set

$$\widetilde{\mathsf{H}} \equiv \{A(T) : F(T) \in \mathsf{H}\}.$$

Theorem 3.1. Denote $S(T) = T^T P_1(T)T$. For any matrices $A \in \mathsf{H}$ and nonsingular T one has
(a) $\lambda_1[(T^T T A)_s S^{-1}(T)] \geq 2\kappa[F(T)]$,
(b) $\mathsf{H}^- \subseteq \widetilde{\mathsf{H}}$,
(c) $A(T) \in \widetilde{\mathsf{H}} \Leftrightarrow \mu[S(T)A] < 0$.
Proof. Having in mind (6) one gets

$$P_1^{-1/2}(T)A(T)P_1^{-1/2}(T) = P_1^{-1/2}(T)F(T)P_1^{1/2}(T) = X.$$

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Since $\mu(X) \geq 2\kappa(X)$ for any matrix X [2] it follows that

$$\mu(X) = \mu[P_1^{-1/2}(T)T^{-1/2}(T)] = \lambda_1[(T^T T A)_s S^{-1}(T)] \geq 2\kappa[F(T)] \Rightarrow (a).$$

Assertion (b) follows immediately, i.e. $A(T) \in \mathsf{H}^- \Leftrightarrow \mu[A(T)] < 0 \Leftrightarrow \mu(T^T T A) < 0 \Leftrightarrow \mu(X) < 0 \Rightarrow F(T) \in \mathsf{H} \Rightarrow A(T) \in \widetilde{\mathsf{H}} \Rightarrow (b).$
Finally, $F(T)$ is unitary and hence normal matrix, or

$$A(T) \in \widetilde{\mathsf{H}} \Leftrightarrow 0 > 2\kappa[F(T)] = \mu[F(T)] = \mu[A(T)P_1^{-1}(T)] \Leftrightarrow \mu[S(T)A] < 0 \Rightarrow (c).$$

Corollary 3.1. For any T, such that $\bar{P} = T^T T$ is an ELM for A, $S(T)$ is also an ELM for A.

Since the main purpose is to get ILM, let $T = I$, i.e. $A(I) \equiv A, F(I) \equiv F,$ $S(I) = P_1(I) \equiv P_1$. Then, the assertions of Theorem 3.1 become:

Corollary 3.2. For any matrix $A \in \mathsf{H}$ one has

(a) $\lambda_1(A_s P_1^{-1}) \geq 2\kappa(F),$

(b) $A \in \mathsf{H}^- \Rightarrow A \in \widetilde{\mathsf{H}},$

(c) $A \in \widetilde{\mathsf{H}} \Leftrightarrow \mu(P_1 A) < 0 \Leftrightarrow \mu(P_2^{-1} A) < 0.$

In other words, $A \in \widetilde{\mathsf{H}}$, if and only if, $(A^T A)^{1/2}$ and $(AA^T)^{-1/2}$ are ILM for A.

Denote $S_1 = (P_1 A)_s$, $S_2 = (P_2^{-1} A)_s$, $\mu_i = \lambda_1(-QS_i^{-1})$, $i = 1, 2$, $R_1 = \mu_1 P_1$, $R_2 = \mu_2 P_2^{-1}$.

Corollary 3.3. Let $A \in \widetilde{\mathsf{H}}$. The solution P in (1) has the following upper matrix bounds

$$(7) \qquad P \leq R_3 = [\alpha R_1 + (1-\alpha)R_2], \ \forall \alpha, \ \alpha \in [0,1].$$

Proof. If $A \in \widetilde{\mathsf{H}}$, then (Corollary 3.2, assertion (c)) one has $0 > S_i \Rightarrow 0 \geq \mu_i S_i + Q \Rightarrow 0 \geq [(R_i - P)A]_s \Rightarrow R_i - P \geq 0, \ i = 1, 2 \Rightarrow [\alpha R_1 + (1-\alpha)R_2] \geq P$, for any $\alpha \in [0,1]$.

Matrix bound (7) is a generalization of the upper estimate presented in [8].

Lemma 3.1. If $A \in \widetilde{\mathsf{H}}$, the maximum eigenvalue and the trace of P in (1) have the following upper bounds:

$$(8) \qquad \lambda_1(P) \leq l_1 = \lambda_1(R_3) \leq \min[\mu_1 \sigma_1(A), \mu_2 \sigma_n^{-1}(A)],$$

$$(9) \qquad \operatorname{tr}(P) \leq t_1 = \min[t_1', t_2', t_1'', t_2'', t],$$

where

$$t_i' = \operatorname{tr}(R_i), \ t_i'' = -\lambda_1^{-1}(S_i D_i^{-2}) \operatorname{tr}(QD_i^{-1}), \ i = 1, 2, \ D_1 = P_1, \ D_2 = P_2^{-1}$$

and

$$t = -\lambda_1^{-1}(S_3 D_3^{-2}) \operatorname{tr}(Q D_3^{-1}), \ S_3 = (D_3 A)_s, \ D_3 = R_3, \ \alpha \in [0, 1].$$

Proof. Consider the matrix bound (7). Then

$$l_1 \leq \alpha\lambda_1(R_1) + (1-\alpha)\lambda_1(R_2) = \tau(\alpha) = \min_{\alpha\in[0,1]} \tau(\alpha) = \min[\lambda_1(R_1), \lambda_1(R_2)],$$

$$\operatorname{tr}(R_3) = \alpha \operatorname{tr}(R_1) + (1-\alpha)\operatorname{tr}(R_2) = \min(t_1', t_2').$$

Consider (1). Having in mind that $S_i < 0$, $i = 1, 2, 3$, one has

$$Q D_i^{-1} = -A^T P D_i^{-1} - P A D_i^{-1}, \ i = 1, 2, 3.$$

Application of the tr operator results in

$$\operatorname{tr}(Q D_i^{-1}) = -2\operatorname{tr}(P A D_i^{-1}) = -\operatorname{tr}[P(A D_i^{-1})_s] = -\operatorname{tr}[P D_i^{-1} S_i D_i^{-1}] \geq -\lambda_1(S_i D_i^{-2})\operatorname{tr}(P)$$

and bounds t, t_i'', $i = 1, 2$ are proved.

The upper trace bound t_1'' was proposed in [7].

The requirement $A \in \widetilde{\mathsf{H}}$ is less restrictive in comparison with the assumption that $A \in \mathsf{H}^-$, due to the fact that $\mathsf{H}^- \subseteq \widetilde{\mathsf{H}}$. Therefore, the presented here bounds (7), (8) and (9) are less conservative with respect to the validity restrictions imposed on matrix A by the existing estimation approaches.

5. Conclusion. Extending the set of Hurwitz matrices for which upper solution bounds for the CALE are valid is the topic of this paper. The available bounds require that $A \in \mathsf{H}^-$ (set of matrices with negative definite symmetric parts), which is a very restrictive condition. Applying an s.v.d. approach, any nonsingular matrix can be represented as a product of a unitary matrix F and a positive definite matrix. It is proved in Theorem 3.1 that

(i) $\mathsf{H}^- \subseteq \widetilde{\mathsf{H}}$ (set of matrices with stable unitary parts F),

(ii) $A \in \widetilde{\mathsf{H}} \Leftrightarrow (A^T A)^{1/2}$ and $(A A^T)^{-1/2}$ are ILM for A.

This helps to get the proposed here upper matrix and scalar solution bounds (Corollary 3.3. and Lemma 3.1.).

The main contribution of this work consists in the attempt made to overcome some well-known difficulties concerning solution bounds for the CALE, such as hard validity restrictions and additional computational burden when the ELM approach to get estimates is used.

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