

VALID INNER UPPER MATRIX BOUND
FOR THE CONTINUOUS LYAPUNOV EQUATION

S. Savov, I. Popchev

(Submitted on August 7, 2006)

Abstract

An always valid upper matrix bound for the solution of the continuous Lyapunov equation is proposed. The estimate is expressed entirely in terms of the equation parameters.

Key words: continuous Lyapunov equation, Hurwitz matrix, solution upper bounds

Introduction. Let X_s denotes the symmetric part of a matrix X , i.e. $X_s = X^T + X$. The continuous algebraic Lyapunov equation (CALE)

$$(1) \quad (PA)_s + Q = 0, \quad Q = Q^T$$

with respect to the unknown matrix P has a positive definite solution for any $Q > 0$, if and only if $A \in H$ (the set of Hurwitz matrices).

Equation (1) plays a fundamental role in modern engineering theory. Due to computational reasons sometimes only a solution estimate is required. The estimation problem for the CALE has attracted considerable attention in the past three decades. Various lower and upper bounds for the eigenvalues, the trace, the determinant and the solution P have been proposed [4]. In practical applications, especially for stability analysis, upper bounds are desired.

In most cases, the available upper bounds are valid under some very conservative assumptions, e.g. $A_s < 0$. This fact inspired the authors to investigate the conditions under which it is possible to loosen these assumptions, or even to get always valid upper bounds. This paper is organized as follows. Some important previous results are presented in Section 2. Section 3 contains the main result – an always valid upper matrix bound is proposed for the first time.

2. Preliminaries. In what follows, the given below notations will be used. The eigenvalues (when real) of $n \times n$ matrix A are denoted by $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$. The singular value decomposition (s.v.d.) of a square nonsingular matrix A is $A = U\Sigma V^T$, $UU^T = VV^T = I$ and $\Sigma = \text{diag}(\sigma_i) > 0$, $i = 1, 2, \dots, n$, and $\sigma_1 = \max\{\sigma_i\}$, $\sigma_n = \min\{\sigma_i\}$.

This work is supported by the Bulgarian Academy of Sciences under grant No 010069/2004.

If $A_s < 0$, the following maximum eigenvalue and trace upper bounds for P in (1) are proposed:

$$(2) \quad \lambda_1(P) \leq l_0 = \lambda_1(-QA_s^{-1}) \quad [5];$$

$$(3) \quad \text{tr}(P) \leq t_0 = - \sum_1^n [\lambda_i(Q)/\lambda_i(A_s)] \quad [3].$$

Let T be a nonsingular matrix, denote $A(T) = TAT^{-1}$ and define the matrix set

$$H^- \equiv \{A(T) : A(T)_s < 0\}.$$

For any given Hurwitz matrix A , $A \notin H^-$, there exists matrix T , such that $A(T) \in H^-$. This well-known fact is used in [1] to apply bound (2), (3) for the modified CALE

$$(4) \quad [\tilde{P}A(T)]_s + \tilde{Q} = 0, \quad \tilde{P} = T^{-T}PT^{-1}, \quad \tilde{Q} = T^{-T}QT^{-1}.$$

Since $A(T) \in H^-$ using Lyapunov's stability argument, it can be shown that

$$(5) \quad P \leq \lambda_1\{-Q[(T^T T A)_s]^{-1}\}T^T T.$$

Using the s.v.d., it is always possible to present A as

$$(6) \quad A = FP_1 = P_2F, \quad F = UV^T, \quad P_1^2 = A^T A, \quad P_2^2 = AA^T.$$

Define the set of matrices with Hurwitz unitary parts, i.e.

$$\tilde{H} \equiv \{A : F \in H\}.$$

Theorem 2.1. For any matrix $A \in H$ one has

- (a) $\lambda_1(A_s P_1^{-1}) \geq 2\kappa(F)$;
- (b) $H^- \subseteq \tilde{H}$;
- (c) $A \in \tilde{H} \Leftrightarrow (P_1 A)_s < 0 \Leftrightarrow (P_2^{-1} A)_s < 0$.

Proof. Having in mind (6), one gets

$$P_1^{-1/2} A P_1^{-1/2} = P_1^{-1/2} F P_1^{-1/2} = X.$$

Since $\lambda_1(X_s) \geq 2\kappa(X)$ [2] for any X , it follows that

$$\begin{aligned} \lambda_1(X) = \lambda_1(A_s P_1^{-1}) \geq 2\kappa(F) &\Rightarrow \text{(a)} \Rightarrow A \in H^- \Leftrightarrow \lambda_1(X_s) < 0 \\ &\Rightarrow \kappa(F) < 0 \Rightarrow A \in \tilde{H} \Rightarrow \text{(b)}. \end{aligned}$$

Finally, F is unitary and hence normal matrix, or

$$A \in \tilde{H} \Leftrightarrow 2\kappa(F) = \lambda_1(F_s) = \lambda_1[(AP_1^{-1})_s] = \lambda_1[(P_2^{-1}F)_s] < 0 \Leftrightarrow \lambda_1[(P_1 A)_s] < 0 \Rightarrow \text{(c)}.$$

The estimation problem for P has three important aspects: (i) validity restrictions on A , (ii) computational burden, and (iii) tightness of the bounds. Bounds based on equation (4) eliminate problem (i), but require the computation of matrix T as a result of some additional procedure and in this sense bound (5) is said to be external matrix

bound (EMB) for P . An internal matrix bound (IMB) is defined entirely in terms of the CALE parameters A and Q , e.g.

$$P \leq B_0 = \lambda_1[-Q(A^T A A_s)^{-1}]A^T A, \quad A \in H^-,$$

$$P \leq B_1 = \lambda_1[-Q(P_1 A)_s^{-1}]P_1, \quad P_1^2 = A^T A, \quad A \in \tilde{H},$$

$$P \leq B_2 = \lambda_1[-Q(P_2^{-1} A)_s^{-1}]P_2^{-1}, \quad P_2^2 = A A^T, \quad A \in \tilde{H},$$

are IMB for P .

Bounds B_1, B_2 can be used to get a generalized IMB

$$P \leq B_g = \alpha_1 B_1 + (1 - \alpha) B_2, \quad \forall \alpha, \alpha \in [0, 1], \quad A \in \tilde{H}.$$

The requirement $A \in \tilde{H}$ is less restrictive in comparison with the assumption $A \in H^-$, since $H^- \subseteq \tilde{H}$.

An always valid upper IMB for P is proposed next for the first time.

3. Main results.

Theorem 3.1. [2] Let $A \circ B$ denotes the Hadamard product of two positive definite matrices. Then $A \circ B$ is also positive definite.

Lemma 3.1. Consider the symmetric Toeplitz matrix $M(\varepsilon) = [m_{ij}]$, $m_{ij} = \varepsilon^{|j-i|}$, $\varepsilon > 0$, $i, j = 1, \dots, n$. Then $M(\varepsilon)$ is positive definite if and only if $\varepsilon < 1$.

Proof. If $M(\varepsilon) > 0$, then by necessity $\varepsilon < 1$. Let $\varepsilon < 1$ and consider the upper triangular matrix

$$L = [l_{ij}], \quad l_{ij} = \begin{cases} \varepsilon^{j-i}; & i = 1, j = 1, n \\ \varepsilon^{j-i}\eta; & i, j = 2, n, i \leq j. \end{cases}$$

Let l_i and l_j , $i \leq j$, define the i -th and j -th column of L , respectively, i.e.

$$l_i = (\tilde{l}_i, 0, \dots, 0)^T, \quad l_j = (\tilde{l}_j, \varepsilon^{j-i-1}\eta, \varepsilon^{j-i-2}\eta, \dots, \varepsilon\eta, \eta, 0, \dots, 0)^T,$$

where

$$\begin{aligned} \tilde{l}_i &= (\varepsilon^{i-1}, \varepsilon^{i-2}\eta, \varepsilon^{i-3}\eta, \dots, \varepsilon\eta, \eta)^T \in R^i, \\ \tilde{l}_j &= (\varepsilon^{j-1}, \varepsilon^{j-2}\eta, \varepsilon^{j-3}\eta, \dots, \varepsilon^{j-i+1}\eta, \varepsilon^{j-i}\eta)^T \in R^i. \end{aligned}$$

Denote $i + j = t$, $j - i = k \geq 0$ and $\eta^2 = 1 - \varepsilon^2 > 0$. Compute the scalar product

$$\begin{aligned} l_i^T l_j &= \tilde{l}_i^T \tilde{l}_j = \varepsilon^{t-2} + \varepsilon^{t-4}\eta^2 + \varepsilon^{t-6}\eta^2 + \dots + \varepsilon^{k+2}\eta^2 + \varepsilon^k \eta^2 \\ &= \varepsilon^{t-2} + \varepsilon^{t-4} - \varepsilon^{t-2} + \varepsilon^{t-6} - \varepsilon^{t-4} + \dots + \varepsilon^{k+2} - \varepsilon^{k+4} + \varepsilon^k - \varepsilon^{k+2} = \varepsilon^k = \varepsilon^{j-i} = m_{ij}. \end{aligned}$$

Therefore, if $\varepsilon < 1$, there exists matrix L , such that $M = L^T L > 0$.

Theorem 3.2. Let $E(\varepsilon) = \text{diag}(e_{ii})$, $e_{ii} = \varepsilon^{n-i+1}$, $i = 1, n$. Denote $R(\varepsilon) = E(\varepsilon)W^*$ and $A(\varepsilon) = R(\varepsilon)AR^{-1}(\varepsilon)$. If $A \in H$, $A \notin H^-$, there exist unitary matrix W and

a) a positive scalar ε_1 , such that $A(\varepsilon_1) \in H^-$, if and only if $\varepsilon_1 < 1$,

b) a scalar $\varepsilon_2 \geq \varepsilon_1$, such that $A(\varepsilon_2) \in \tilde{H}$.

Proof. Any square matrix is unitary similar to an upper triangular matrix, i.e.

$$A = W T W^*, \quad W W^* = I, \quad T = -\Lambda + T_0,$$

where $T_0 = [t_{ij}]$, $t_{ij} = 0$ for $i \geq j$ and Λ is a diagonal matrix.

Then, $A(\varepsilon)$ is an upper triangular matrix defined as

$$A(\varepsilon) = -\Lambda + T_0(\varepsilon), \quad T_0(\varepsilon) = [t_{ij}(\varepsilon)], \quad t_{ij}(\varepsilon) = \varepsilon^{j-i} t_{ij}.$$

For any ε , one has

$$A(\varepsilon) \in H^- \Leftrightarrow \Lambda_s - T_{0s}(\varepsilon) > 0.$$

Under the assumptions $A \in H$, $A \notin H^-$, it follows that $\Lambda_s > 0$ and $-A_s(1) = \Lambda_s - T_{0s}(1) = \Lambda_s - T_{0s}$ is not a positive definite matrix. Suppose that there exists some ε , $0 < \varepsilon < 1$, such that $-A_s(\varepsilon^{-1}) > 0$. Matrix $-A_s(1)$ can be presented as a Hadamard product of two symmetric matrices, i.e.

$$-A_s(1) = -A_s(\varepsilon^{-1}) \circ M(\varepsilon),$$

where $M(\varepsilon)$ is defined in Lemma 3.1 and according to it is positive definite for any $0 < \varepsilon < 1$. Then, $-A_s(\varepsilon^{-1}) > 0$ and $M(\varepsilon) > 0 \Rightarrow -A_s(1) > 0$, which contradicts the assumption $A \notin H^-$. Therefore, if $A \notin H^-$, there does not exist $\varepsilon > 1$, such that $A(\varepsilon) \in H^-$. Obviously, there exists some sufficiently small $\varepsilon_1 < 1$, such that $A(\varepsilon_1) \in H^-$. Since, $H^- \subseteq \tilde{H}$, if $A \in \tilde{H} \Leftrightarrow A(1) \in \tilde{H}$, then $\varepsilon_1 < 1 = \varepsilon_2$. Otherwise, whenever $A(\varepsilon_1) \in H^- \Rightarrow A(\varepsilon_1) \in \tilde{H} \Rightarrow \varepsilon_1 \leq \varepsilon_2$.

Corollary 3.1. A given matrix A is Hurwitz, if and only if there exists positive scalar ε , such that the following conditions hold:

- 1) $[R^*(\varepsilon)R(\varepsilon)A]_s < 0$;

- 2) $\{[\alpha R_1(\varepsilon) + (1-\alpha)R_2(\varepsilon)]A\}_s < 0$, $\alpha \in [0, 1]$, where $R_1^2(\varepsilon) = A^*(\varepsilon)A(\varepsilon)$, $R_2^{-2}(\varepsilon) = A(\varepsilon)A^*(\varepsilon)$ and matrices $R(\varepsilon)$ and $A(\varepsilon)$ are defined in Theorem 3.2.

Since matrices $R(\varepsilon)$, $R_1(\varepsilon)$ and $R_2(\varepsilon)$ are determined entirely in terms of matrix A , they can be used to get upper IMB of kind (5) for CALE. Such result is presented for the first time.

5. Conclusion. The main contribution of this research work consists in defining an appropriate nonsingular transformation for the coefficient matrix A , such that it is always possible to define an upper matrix bound for CALE, whenever a positive definite solution exists. The proposed new estimate is expressed entirely in terms of the equation's parameters.

In other words, an attempt is made to overcome some well-known difficulties concerning solution bounds for CALE, such as hard validity restrictions and additional computational burden when ELM approach to get estimates is used.

REFERENCES

- [1] FANG Y., K. LOPARO, X. FENG. IEEE Trans. on Automatic Control, **42**, 1997, No 3, 408–411. [2] HORN R., C. JOHNSON. Topics in matrix analysis, Cambridge, Cambridge University Press, 1991. [3] KOMAROFF N. IEEE Trans. on Automatic Control, **37**, 1992, No 5, 337–341. [4] KWON W., Y. MOON, S. AHN. Int. J. Control, **64**, 1996, No 3, 377–389. [5] YASUDA K., K. HIRAI. IEEE Trans. on Automatic Control, **24**, 1979, No 6, 483–487.

*Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 2
1113 Sofia, Bulgaria*